Chapter 13

Sequences & Series

Sequence, Series and Series expansion

By now, any students in this level of Calculus have seen the concept of Sequences and Series in algebra or pre-Calculus classes. I am not going to repeat all the information that you already know. This is a new way of looking at the Sequences and Series. It preserves the main fundamental idea of calculus which is continuity of Space and time to make a meaningful reason why we are covering Sequences and Series in a Calculus class, plus also why we cover Sequence and Series in absence of complex space.

Think of Sequences (a_n) as a function with an independent variable which is element of Natural numbers $(Domain : \aleph)$ and each value of the Sequence is element of Complex numbers. Almost all properties of functions in Calculus can be applied to Sequences by keeping in mind that the sequences are not continues functions due the fact that $n \in \aleph$.

Series $(\sum a_n)$ are sum of the Sequences (a_n) and it analogous to evaluating an integral in Calculus. So, the sum operator similar to integral operator is also a linear operator.

$$L[af(x) + bg(x)] = aL[f(x)] + bL[g(x)]$$

The Sequences similar to functions can approaches to a Number (It is called Converging Sequence) or does not go to a Number (it is called Diverging Sequence).

The Series similar to evaluation of an integral can approaches to a Number (It is called Converging Series) or does not go to a Number (it is called Diverging Series).

In almost all the cases in application it is desire for series to converge if it doesn't then sometimes there are techniques to make them converge.

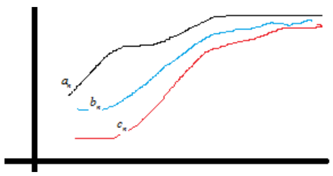
In this section we will look at 9 different tests for convergent of a Series. Unfortunately (sadly) there will be not much of proofs. Mostly we base them on logic and intuition. Lots of the proofs must be done in Complex space which we haven't study yet.

I will wave my hands on some of these proofs to give some sense to them. Let's start:

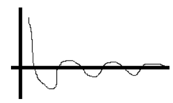
Definition: If the Sequence approaches to a number then it is a converging Sequence. The underline a number has two properties. a) It is a single number. b) It is a finite number.

$$\lim_{n\to\infty} a_n = A\ Number$$

Squeeze theorem: (Without proof) if Sequences $a_n \le b_n \le c_n \ \forall n \ge n_0$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} c_n = L$ then they squeeze $\lim_{n\to\infty} b_n = L$



Theorem: (Without proof) If $\lim_{n\to\infty} |a_n| = 0 \Rightarrow \lim_{n\to\infty} a_n = 0$



Class work set #1

Indicate if the Sequence is Converges or Diverges, if it converges find the limit.

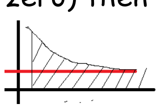
$a)\ \lim_{n\to\infty}(1+\frac{2}{n})^{3n} =$ $\quad b)\ \lim_{n\to\infty}(Tan^{-1}n) =$ $\quad c)\ \lim_{n\to\infty} Tanh(n) =$

$d)\ \lim_{n\to\infty}\frac{(-1)^n(n^2-1)}{n^2} =$ $\quad e)\ \lim_{n\to\infty} Sin^{-1}(n) =$ $\quad f)\ \lim_{n\to\infty} Cosh(\frac{1}{n}) =$

$g)\ \lim_{n\to\infty} nSin(\frac{1}{n}) =$ $\quad i)\ \lim_{n\to\infty} e^{-n} =$ $\quad j)\ \lim_{n\to\infty}\frac{Sin(n)}{n} =$

Series and Sequences have different rules since the sum of a sequence is Series then to find whether the Series converge /diverge, one must apply some tests on them.

Zero Test #1: If the Sequence of a Series does not approach to Zero then it is a Diverging Series. $\lim_{n\to\infty} a_n \ne 0 \ \Rightarrow \sum a_n$ is Diverge. If a function approaches to a limit (not zero) then the area under the curve increases larger and larger as the x goes to infinity.



Geometric Series Test#2: Geometric Series are in the form of $\sum a_1(r)^{n-1}$ where n starts from one. In algebra class we have seen the sum is $\sum_{1}^{N} a_1(r)^{n-1} = \frac{a_1(1-r^N)}{(1-r)}$ and if $0<|r|<1$

then the sum will be a finite number. If the Series has the form of a Geometric Series with ratio less than One, then check the ratio (r). $\begin{cases} 0<|r|<1 & Converges \\ |r|\geq 1 & Diverges \end{cases}$.

If any number ($-1<x<1$) raised to $n>1$ the result is smaller than the number. $x^n<x$

Hyper harmonic Series Test #3:are in the form of $\sum \frac{1}{n^p}$ where n starts from one.

If the Series has the form of a hyper harmonic Series, then check the exponent (p).

$\begin{cases} P>1 & Converges \\ P\leq 1 & Diverges \end{cases}$ In case $P=1$ the Series is called Harmonic Series

Telescopic Test #4: Telescopic method is when the series is expanded and the terms cancel so the series reduces to a limit.

$\sum a_n$	$\sum b_n$	$\sum(a_n+b_n)$	$\sum(a_n-b_n)$
Converge	Converge	Converge	Converge
Diverge	Diverge	Diverge	Need a test
Converge	Diverge	Diverge	Diverge
Diverge	Converge	Diverge	Diverge

Class work set #2

Indicate if the Series is Converges or Diverges. (Use Test zero to Test #4)

$a)\ \sum_{1}^{\infty}(1-\frac{1}{n})^n=$ $b)\ \sum_{2}^{\infty}(\frac{1}{n^2-1})=$ $c)\ \sum_{1}^{\infty}Tanh(n)=$

$d)\ \sum_{1}^{\infty}\frac{Sin(n)}{n^2}=$ $e)\ \sum_{1}^{\infty}\frac{Sin^{-1}(\frac{1}{n})}{Sin(\frac{1}{n})}=$ $f)\ \sum_{1}^{\infty}Cosh(\frac{1}{n})=$

$g)\ \sum_{1}^{\infty}nSin(\frac{1}{n})=$ $i)\ \sum_{1}^{\infty}\frac{e^n}{2^n}=$ $j)\ \sum_{1}^{\infty}\frac{n}{n+2}=$

$k)\ \sum_{1}^{\infty}2(\frac{1}{3})^{2n}=$ $l)\ \sum_{1}^{\infty}(\frac{e}{\pi})^{n-1}=$ $m)\ \sum_{1}^{\infty}Ln(\frac{n-1}{n+1})=$

$n)\ \sum_{1}^{\infty}(\frac{1}{n})^{\frac{3}{2}} =$ $o)\ \sum_{1}^{\infty}10^{10}(\frac{10^{10}}{n}) =$ $p)\ \sum_{1}^{\infty}\frac{Sin^2(n)}{n^2} =$

Ratio Test #5: Let $\sum a_n$ be a positive term series then Ratio test gives the following information about the series $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \begin{cases} r > 1 & Diverges \\ r = 1 & No\ Conclusion \\ r < 1 & Converges \end{cases}$

Root Test #6: Let $\sum a_n$ be a positive term series then Root test gives the following information about the series $\lim_{n\to\infty}\sqrt[n]{a_n} = \begin{cases} R > 1 & Diverges \\ R = 1 & No\ Conclusion \\ R < 1 & Converges \end{cases}$

Integral Test #7: Let $\sum a_n$ be change to $\int_a^{\infty} f(x)dx$. Integral test gives the following information about the series. $\int_a^{\infty} f(x)dx = \begin{cases} Infinite\ Area \to Diverges \\ Finite\ Area \to Converges \end{cases}$

Class work set #3

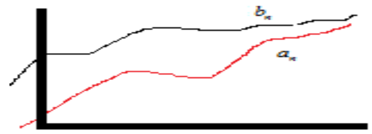
Indicate if the Series is Converges or Diverges. (Use Test #5 to Test #7)

$a)\ \sum_{1}^{\infty}\frac{6+2^n}{4+3^n} =$ $b)\ \sum_{2}^{\infty}\frac{(n+2)!}{(n-1)!\ 2^n} =$ $c)\ \sum_{1}^{\infty}\frac{1}{n(n+2)} =$

$d)\ \sum_{1}^{\infty}nTan^{-1}(n^2) =$ $e)\ \sum_{1}^{\infty}\frac{e^n}{e^{2n}-1} =$ $f)\ \sum_{1}^{\infty}(\frac{n-1}{n^2})^n =$

$g)\ \sum_{1}^{\infty}\frac{(2n+5)!}{(3n)!} =$ $i)\ \sum_{1}^{\infty}ne^{-n} =$ $j)\ \sum_{2}^{\infty}\frac{n}{n^2-1} =$

Comparison Test #8: Let $\sum a_n$ and $\sum b_n$ be a positive term series.



i) If $a_n \le b_n$ and $\sum b_n$ Converges, then $\sum a_n$ Converges

ii) If $a_n \le b_n$ and $\sum a_n$ Diverges, then $\sum b_n$ Diverges

Limit Comparison Test #9: Let $\sum a_n$ and $\sum b_n$ be a positive term series.

For a known a_n if $\lim_{n\to\infty} \frac{a_n}{b_n} =$ A non-zero number then a_n *and* b_n are either Convergent or both Divergent.

Class work set #4

Indicate if the Series is Converges or Diverges. (Use Test #7 to Test #8)

$a)\ \sum_{1}^{\infty} \frac{Sin^2(n)}{n^2} =$ $\qquad b)\ \sum_{2}^{\infty} (\frac{1}{n^2-1}) =$ $\qquad c)\ \sum_{1}^{\infty} \frac{1}{n(n+2)} =$

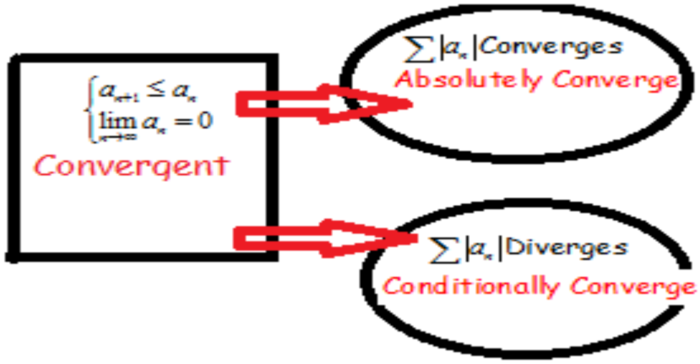
$d)\ \sum_{1}^{\infty} \frac{Sin(n)}{n^2} =$ $\qquad e)\ \sum_{1}^{\infty} \frac{6+2^n}{4+3^n} =$ $\qquad f)\ \sum_{2}^{\infty} \frac{n}{n^2-2} =$

$g)\ \sum_{1}^{\infty} nSin(\frac{1}{n}) =$ $\qquad h)\ \sum_{1}^{\infty} \frac{e^n}{2^n} =$ $\qquad i)\ \sum_{2}^{\infty} \frac{n+2}{n^2-1} =$

Alternative series

If the terms of a series alternate in sign then the series is called Alternating Series. It is represented as $\sum_{0}^{N} (-1)^n a_n$.

- For an alternative Series to converge, two conditions must hold. $\begin{cases} a_{n+1} \le a_n \\ \lim_{n\to\infty} a_n = 0 \end{cases}$
- If $\sum |a_n|$ Converges then the Series Absolutely Converges
- If $\sum |a_n|$ Diverges then the Series Conditionally Converges



Otherwise the Series Diverges.

Class work set #5

a) $\sum_{1}^{\infty} \frac{(-1)^n}{n^{3/2}+1}$ b) $\sum_{1}^{\infty} \frac{(-1)^n}{(n+1)!}$ c) $\sum_{1}^{\infty} \frac{(-1)^n (n^2-5)}{(2n-1)}$ d) $\sum_{1}^{\infty} \frac{(-1)^n \sqrt[4]{n}}{n-5}$

e) $\sum_{1}^{\infty} \frac{(-1)^n}{2n\sqrt{\ln(n)}}$ f) $\sum_{1}^{\infty} \frac{n^n}{(-2)^n}$ g) $\sum_{1}^{\infty} \frac{(-1)^n}{(n-1)^2+1}$ h) $\sum_{1}^{\infty} \frac{(-1)^{n+1}}{3n+4}$

Radius of Convergent

On Graph of $y = \frac{1}{x^2-1}$, there are two vertical asymptotes at $x=-1,\ x=1$. At these points the function is not defined and its value goes to infinite nearby these points. These points are Singularity of the function. Any other point on the function gives a finite value so the function (Sequence) is Converging. Radius of Convergent is the distance from a given point to the nearest Singularity. For the point at $x = 0.9$ then the radius of Convergent is 0.10 to $x=1$. And at point $x=-0.4$ is 0.60 to $x=-1$.

Why do we call this distance a radius? On Graph of $y = \frac{1}{x^2+1}$, there are two Singularities at $x=-i,\ x=i$ in Complex plane. This distance is radius of a ball in Complex N- Dimension.

Class work set #6

a) $\sum_{1}^{\infty} \frac{(3x-1)^n}{2^n}$ b) $\sum_{1}^{\infty} \frac{(x-4)^n}{n}$ c) $\sum_{1}^{\infty} \frac{2^n (x-1)^n}{(3n+1)}$ d) $\sum_{1}^{\infty} \frac{(2x)^n}{(n+1)!}$

e) $\sum_{1}^{\infty}\frac{(x-2)^n}{4^n}$ f) $\sum_{1}^{\infty}\frac{(-1)^n(2x-3)^n}{(n+2)^2}$ g) $\sum_{1}^{\infty}\frac{(2n)!}{(n+1)^3}(x-1)^n$ h) $\sum_{1}^{\infty}\frac{(-1)^n(3x+1)^n}{(n+1)}$

Taylor Series expansion

Taylor Series was introduced by Brook Taylor in 1715, although it was known to James Gregory at least 45 years earlier. Its importance was not fully recognized until 1755 when Euler applied it in his development of differential calculus. The proof of the "Taylor Series" starts with the Fundamental Theory of Calculus.

$\int_a^z f'(x)dx = f(x) - f(a)$ so, $f(x) = f(a) + \int_a^z f'(x)dx$ similarly we have

$f'(x) = f'(a) + \int_a^z f''(x)dx$ So, f(x) is $f(x) = f(a) + \left[\int_a^z f'(a) + \int_a^z f''(x)dx\right]dx$

$f(x) = f(a) + f'(a)(x-a) + \int_a^z\int_a^z f''(x)dxdx$ ←Double integral

Repeating the same concept $f''(x) = f''(a) + \int_a^z f'''(x)dx$

$$f(x) = f(a) + f'(a)(x-a) + \int_a^z\left[\int_a^z f''(a) + \int_a^z f'''(x)dx\right]dxdx$$

$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \int_a^z\int_a^z\int_a^z f'''(x)dxdxdx$ ←Triple integral

Repeating this process for some more terms we have the Taylor expansion

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + ... + \frac{f^{n-1}(a)(x-a)^{n-1}}{(n-1)!} + R_n$$

The above formula allows you to find Taylor expansion of (almost) any function near a point.

For the given function $f(x)$

1- Take the first few differentiation of the function.

2- Evaluate the derivatives at point "a"

3- Use the above values and value of "a" to write out the expansion

4- If it is possible to write the expansion in a compact form (Sun notation)

You have seen linearization in Cal 1A which is the first two terms of expansion

Isaac Newton used the linearization of a function for a small change in distance compare to large distances measured, so he did not need to expand the function to more than two terms of expansion. Others took the concept of expansion and extended to more terms

Classwork set #7

Find the "Taylor Series" of the following functions at $x = a = 0$ (Maclaurin)

$a)\ y = e^x$ $\quad b)\ y = Cosx$ $\quad c)\ y = Sinx$ $\quad d)\ y = xSinx$

$e)\ y = \dfrac{1}{1-x}$ $\quad f)\ y = \dfrac{1}{1+x^2}$ $\quad g)\ y = Ln(1-x)$ $\quad h)\ y = Tan^{-1}x$

Let's look at the expansion of $y = e^x$. We have $e^x \approx 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \bullet\bullet\bullet + \frac{x^n}{n!} = \sum_0^n \frac{x^n}{n!}$

That means $e^x \approx 1$(at zero) and $e^x \approx 1 + \frac{x}{1!}$ (at ε away vicinity of zero) $e^x \approx 1 + \frac{x}{1!} + \frac{x^2}{2!}$ as the point gets farther. And So on.

Other ways of finding Taylor expansion are as follows: a) Decomposition b) Substitution c) Integration d) Differentiation e) combination

Decomposition

We have seen in Cal 1B that $e^{ix} = Cos(x) + iSin(x)$ and in pre Calculus courses we showed that $Cos(x)$ is an even function, where as $Sin(x)$ is an odd function

$$e^{ix} \approx 1 + i\frac{x}{1!} - \frac{x^2}{2!} - i\frac{x^3}{3!} + \frac{x^4}{4!} + i\frac{x^5}{5!} - \frac{x^6}{6!} \bullet\bullet\bullet + \frac{(i)^n x^n}{n!}$$

Re-Collecting the real parts and imaginary parts in to two sets

$$e^{x} \approx [1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \bullet\bullet\bullet + \frac{(-1)^n x^{2n}}{(2n)!}] + i[\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} \bullet\bullet\bullet + \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!}] =$$

$$e^{x} \approx \sum_{0}^{n} \frac{(-1)^n x^{2n}}{(2n)!} + i\sum_{0}^{n} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!}$$ Then we get the expansion for $Cos(x), Sin(x)$

$$Cos(x) \approx \sum_{0}^{n} \frac{(-1)\ \ x^{2n}}{(2n)!} \quad and \quad Sin(x) \approx \sum_{0}^{n} \frac{(-1)^{n+1} x^{2n+1}}{(2n+1)!}$$

Substitution

We have seen in Pre Cal that $\sum_{0}^{n} x^n = \frac{1}{1-x}$ For small x. So, to find Series expansion of

a) $\frac{1}{1+x}$ Use Substitution $x \to -x$ then $\frac{1}{1+x} = \sum_{0}^{n} (-1)^n x^n$

b) $\frac{1}{1-x^2}$ Use Substitution $x \to x^2$ then $\frac{1}{1-x^2} = \sum_{0}^{n} x^{2n}$

c) $\frac{1}{1+x^2}$ Use Substitution $x \to -(x^2)$ then $\frac{1}{1+x^2} = \sum_{0}^{n} (-1)^n x^{2n}$

Integration

We have seen in Cal 1B that $\int \frac{1}{1+x} dx = Ln(1+x) + c$. To find Series expansion of

a) $Ln(1+x)$ Use integrate of $\frac{1}{1+x}$ then $Ln(1+x)=\sum_{0}^{n}\frac{(-1)^n}{n+1}x^{n+1}$

b) $Ln(1-x)$ Use integrate of $\frac{1}{1-x}$ then $Ln(1-x)=-\sum_{0}^{n}\frac{1}{n+1}x^{n+1}$

c) $Tan^{-1}x$ Use integration of $\frac{1}{1+x^2}$ then $Tan^{-1}(x)=\sum_{0}^{n}\frac{(-1)^n}{2n+1}x^{2n+1}$

Differentiation

We have seen in Cal 1B that $\frac{d}{dx}Sinx=Cosx$. To find Series expansion of

a) $Cosx$ Use differentiation of $Sinx$ then $Cosx=\frac{d}{dx}\sum_{0}^{n}\frac{(-1)^{n+1}x^{2n+1}}{(2n+1)!}=\sum_{0}^{n}\frac{(-1)\ x^{2n}}{(2n)!}$

b) $\frac{1}{(1+x)^2}$ Use differentiation of $\frac{1}{1+x}$ then $-\frac{d}{dx}\sum_{0}^{n}(-1)^n x^n=\sum_{1}^{n}n(-1)^{n+1}x^{n-1}$

Combinatiation

a) $xTan^{-1}x$ Use product of x and $Tan^{-1}(x)$ then $xTan^{-1}(x)=\sum_{0}^{n}\frac{(-1)^n}{2n+1}x^{2n+2}$

b) $xSinx$ Use product of x and $Sinx$ then $xSinx=\sum_{0}^{n}\frac{(-1)^{n+1}x^{2n+2}}{(2n+1)!}$

The series expansion of a function can be used to find limit of a function or evaluation of integral (in a very small interval nearby the desire point).

Let's look at two examples

$$a) \lim_{x\to 0}\frac{1-Cosx+Sinx}{x}=\lim_{x\to 0}\frac{1-(1-\frac{x^2}{2}+\bullet\bullet\bullet)+(x-\frac{x^3}{3!}+\bullet\bullet\bullet)}{x}=\frac{1-1+x}{x}=1$$

$$b) \int_0^{0.1} x\, e^{-x}dx=\int_0^{0.1}(x-x^2)dx=(\frac{x^2}{2}-\frac{x^3}{3})/_0^{0.1}$$

Application

The value of $e, \pi, and\ i$ are the most interesting values to mathematics and Sciences.

Let's find the estimation of each. We have the Series expansion

$e^x=1+x+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4+\frac{1}{5!}x^5+...$ then $e^1=1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+...+\frac{1}{8!}+\frac{1}{9!}\approx 2.71828153$

$$\tan^{-1}x=x-\frac{1}{3}x^3+\frac{1}{5}x^5+...$$

$$\pi=4\tan^{-1}(1)=4(1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+...)$$

Unfortunately this series converges too slowly to be used for computation.

This is a better Series $\pi=4(\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3})$.This was introduced by Leibniz in 1674.

Examples-Approximate $\ln(1.2)$ and $Sin(0.1)$

$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+..$ $\quad\quad$ $\ln(1+0.2)=0.2-\frac{(0.2)^2}{2}+\frac{(0.2)^3}{3}-\frac{(0.2)^4}{4}+...$

Approximation with only 4 terms $\ln(1.2)\cong 0.1822666$ which has an error of 18.232%

$Sinx=x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+...$ $\quad\quad$ where x = 0.1

Practice Estimate $Sin(46^\circ), Cos(0.1), \ln(0.9)$

Note: in the case of angles, we have to evaluate them in Radians

We have work with Hook's law $\vec{F} = -k\vec{x}$ and potential energy of Spring $PE = (1/2)kx^2$

Potential Energy for spring is a function of x, we can expand the function $f(x)$ by Taylor Series. $f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2!} f''(x_0)(x - x_0)^2 + \ldots$

Maximum potential energy is when $f'(x_0) = 0$ and all the energy is potential, take $f(x_0)$ as origin so that $f(x_0) = 0$. Then we left with $f(x) \cong \frac{1}{2} f''(x_0)(x - x_0)^2$ since the second derivative is constant at $x = x_0$ ($f''(x_0) = k$) then the Potential Energy is $PE = \frac{1}{2}kx^2$.

Solving Differential equations

Solve $y' = 1 - xy$ at point (1,0)

$x = 1 \qquad y = 0 \qquad y' = 1$

$$y'' = -y - xy' = -1$$
$$y''' = -2y' - xy'' = -1$$
$$y'''' = -3y'' - xy''' = 4$$
$$y(x) \approx 0 + 1(x-1) + \frac{-1}{2!}(x-1)^2 + \frac{-1}{3!}(x-1)^3 + \frac{4}{4!}(x-1)^4 + \ldots$$

Solve the following Diff. Eq Classwork #8

$a)\ y' = x + 2y\ at\ (0,1)$

$b)\ y' = x + 2y\ at\ (1,0)$

$c) y' = yx\ at\ (0,0)$