Chapter 12

Table of Contents

Chapter 12 1

Table of Contents 1

Chapter 12- Curved space vector Calculus 2

Homework set 1 (12 questions) 3

Continuity 4

To find the tangent line to a curve: 5

Method 1: Evaluate the vector function at different values of t (time). 6

Method 2: Eliminate the parameter "t" and write the equation in terms of x and y only 7

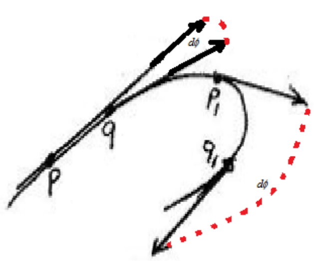
Unit Normal vector and Binormal vector 7

Example: $\vec{r}(t) =< t,\ t^2 > \quad \Rightarrow \quad \vec{r}\,'(t) =< 1,\ 2t > \quad \Rightarrow \quad \vec{r}\,''(t) =< 0,\ 2 >$ 7

Tangent plane, Normal plane, and osculating plane 8

Homework set 2 (6 questions) 8

Curvature 8



Space curvature creates acceleration 9

So $\vec{r}\,''(t) = a_T\vec{T} + a_N\vec{N}$ Equation of acceleration involves curvature 9

Let's find curvature 9

Homework Set 3 (5 questions) 10

Some basic parametrized curved 11

a) Given $\vec{r}(t) =< \frac{2}{1+t^2} - 1,\ \frac{2t}{1+t^2} >$ and $\vec{r}\,'(t) =< \frac{-4t}{(1+t^2)^2},\ \frac{2-2t^2}{(1+t^2)^2} >$therefor $|\vec{r}\,'(t)| = \frac{2}{1+t^2}$ 12

c) Replace t by function of s in $\vec{r}(t)$. Use trig identities to get $\vec{r}(s) =< Cos(s),\ Sin(s) >$ 12

Homework Set 4 (7 questions) 12

Show that Exercises 13

2- Show that $\kappa = \left|\frac{d\phi}{ds}\right|$ $\quad Given\ \vec{T} = T < Cos\phi, Sin\phi >$

$\kappa = \left|\frac{d\vec{T}}{ds}\right| = \left|\frac{d\vec{T}}{d\phi}\frac{d\phi}{ds}\right| = \left|\frac{d\vec{T}}{d\phi}\right|\left|\frac{d\phi}{ds}\right| = \left|< -Sin\phi,\ Cos\phi >\right|\left|\frac{d\phi}{ds}\right| = \left|\frac{d\phi}{ds}\right|$ 13

3- Show that $\frac{d\vec{B}}{ds} \perp \vec{B}$ 13

4- Show that $\frac{d\vec{B}}{ds} \perp \vec{T}$ 13

5- Show that $\frac{d\vec{B}}{ds} = -\tau_{(s)}\vec{N}$ Where $\tau_{(s)}$ is torsion of the curve 14

6- Show that for a plane carve $\tau_{(s)} = 0$ 14

7- Show that $\frac{d\vec{N}}{ds} = \tau_{(s)}\vec{B} - \kappa\vec{T}$ 14

Vectorial Mechanics 14

Dynamic of rotations 15

Torque is cross product of moment arm and applied force $\vec{\Gamma}(t) = \vec{r}(t) \times \vec{F}(t)$ 15

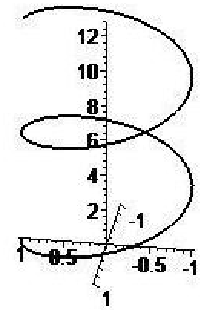
Given $\vec{F}(t) = C(t)\vec{r}(t)$ where $C(t)$ is a scalar function 15

Homework set 5 (6 questions) 15

Chapter 12- Curved space vector Calculus

We have defined vector function $\vec{r}(t) =< x(t), y(t), z(t) >$ where the components are function of time (Parametrized "t") and increasing the parameter "t" determines the orientation of the Curve C.

Example: Graph of Curve $C:\vec{r}(t) = < aCos(t), aSin(t), bt >$ is a circular helix. If the curve has smooth parametrization and it does not intersect itself then the arc length can be



calculated in 3D Space as follows $L = \int_a^b \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}\ dt$

Practice: Find the arc length of the above circular helix in terms of a, b for interval [0,1].

Homework set 1 (12 questions)

Problem: Find the arc length of the parameterized curves below.

1. $x(t) = 5t, \quad y(t) = 4t^2, \quad z(t) = 3t^2, \quad 0 \le t \le 2$

2. $\vec{r}(t) = < t^2, tSin(t), tCos(t) > \qquad 0 \le t \le 1$

3. $\vec{r}(t) = < e^t Cos(t),\ e^t,\ e^t Sin(t) > \qquad 0 \le t < 2\pi$

Problem: Sketch the curve C determined by $\vec{r}(t)$ and indicates the direction (orientation) of increasing time.

4) $\vec{r}(t) = < e^t Cos(t),\ e^t Sin(t) > \quad 0 \le t < \pi$

5) $\vec{r}(t) = < t,\ 2t^2,\ 3t^3 > \quad \forall t \in \Re$

6) $\vec{r}(t) = < 2Cosh(t),\ 3Sinh(t) > \quad \forall t \in \Re$

7) $\vec{r}(t) = < t^3,\ t^2,\ t > \quad 0 \le t \le 4$

8) $\vec{r}(t) = < t^2 + 1,\ t,\ 3 > \quad \forall t \in \Re$

9) $\vec{r}(t) = < 6Sin(t),\ 4,\ 25Sin(t) > \quad [-2\pi, 2\pi)$

10) $\vec{r}(t) = < t,\ t,\ Sin(t) > \quad \forall t \in \Re$

11) $\vec{r}(t) = < t,\ 2t,\ e^t > \quad \forall t \in \Re$

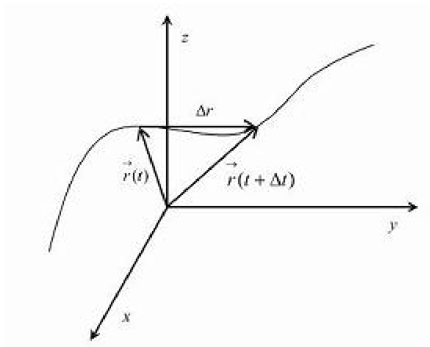
12- a) Show that a twisted cubic $\vec{r}(t) = < at,\ bt^2,\ ct^3 > \quad \forall t \in \Re^+$ intersect any given plane in at most three points.

b) Find the length of the twisted cubic $\vec{r}(t) =< 6t\ \ 3t^2,\ t^3 > \quad 0 \le t \le 1$

In classical Physics, to know the equation of motion of a particle, two properties of a particle are needed. These two are *Position* and *Velocity* of the particle. Classically one can measure both of them at the same time.

If $\vec{r}(t) =< f(t),\ g(t),\ h(t) >$ is position of particle at any time, then with a very small change Δt the position vector changes to $\vec{r}(t+\Delta t) =< f(t+\Delta t),\ g(t+\Delta t),\ h(t+\Delta t) >$ so change in position is $\Delta\vec{r}(t) = \vec{r}(t+\Delta t) - \vec{r}(t)$ and the average speed is $\dfrac{\vec{r}(t+\Delta t) - \vec{r}(t)}{(\Delta t)}$.

Instantaneous Speed: The limit of average Speed as the change in time goes to zero



$$\vec{V}(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{(\Delta t)} = \vec{r}\,'(t)$$

Continuity

The Vector function $\vec{r}(t)$ is said to be continues at $t = t_0$ if, to each $\varepsilon > 0$, there corresponds a $\delta > 0$ such that $|\vec{r}(t) - \vec{r}(t_0)| < \varepsilon \Leftrightarrow |t - t_0| < \delta$

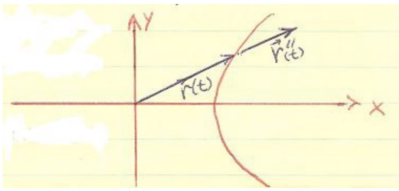
If $\vec{r}(t) =< f(t),\ g(t),\ h(t) >$ then velocity vector and acceleration vector are

$\vec{v}(t) = \vec{r}\,'(t) =< \dfrac{d}{dt} f(t),\ \dfrac{d}{dt} g(t),\ \dfrac{d}{dt} h(t) >$ and $a(t) = \vec{r}\,''(t) =< \dfrac{d^2}{dt^2} f(t),\ \dfrac{d^2}{dt^2} g(t),\ \dfrac{d^2}{dt^2} h(t) >$

Example: $\vec{r}(t) =< rCosh(\omega t),\ rSinh(\omega t),\ 0 >$ is a Hyperbola with $\vec{v}(t)$ and $\vec{a}(t)$

$\vec{r}\,'(t) =< \omega r Sinh(\omega t),\ \omega r Cosh(\omega t),\ 0 >$, $\vec{r}\,''(t) =< \omega^2 r Cosh(\omega t),\ \omega^2 r Sinh(\omega t),\ 0 >= \omega^2 \vec{r}(t)$

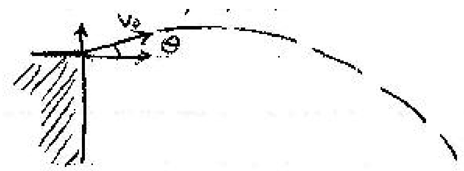
Since $\vec{r}\,''(t) = \omega^2 \vec{r}(t)$ Then acceleration and position are in the same direction.



Since force is product of mass and acceleration for constant mass, then $\vec{F} = m\vec{r}(t)\omega^2$

Projectile motion: If acceleration and initial velocity and position are given, one can find the velocity and position vector by simple integration. Let's turn the axes in a direction so that the particle moves on the x-y plane. Then $\vec{F} = m\vec{a}$ and $\vec{a} =< 0, -g > m/s^2$

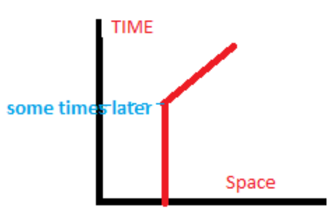
$\vec{v} =< (vCos\theta),\ -gt + (vSin\theta) >$and $\vec{r}(t) =< (vCos\theta)t + x_0,\ \frac{-g}{2}t^2 + (vSin\theta)t + y_0 >$



Unit tangent vector to a curve: Unit tangent vector is the direction velocity of the particle at a given time. It can be calculated $\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$

Smooth Curves: If the speed of a particle is zero at $t_0 = 0$ then the particle is at rest. If $\vec{r}'(t_0) = 0$and $\vec{r}'(t) \neq 0$then curve $\vec{r}'(t)$is not a smooth curve.

It means that if the velocity is zero at some time then the world line is vertical in space-time diagram. If at time "t" the speed is not zero, then the world line changed direction and it is not smooth curve.



To find the tangent line to a curve:

1- Find $\vec{r}'(t)$and then unit tangent vector $\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$

2- At the given point, find time "t".

3- Use "t" to find $\vec{T}(t)$ as the directional vector and write the equation of tangent line.

Note: To solve a problem in Mathematics you can use $\vec{r}\,'(t)$ for directional vector.

Example: Find the tangent line to a parametric equation $\vec{r}(t) = <2t^3 - 1,\ -5t^2 + 3,\ 8t + 2>$ at point $P(1,\ -2,\ 10)$.

Step 1- $\vec{r}\,'(t) = <6t^2,\ -10t,\ 8>$

Step 2- $2t^3 - 1 = 1 \quad -5t^2 + 3 = 2 \quad 8t + 2 = 10 \quad t = 1\,\text{sec}$

$\vec{r}\,'(1) = <6,\ -10,\ 8>$ and $\vec{T}(1) = \dfrac{<6,-10,8>}{\sqrt{200}}$

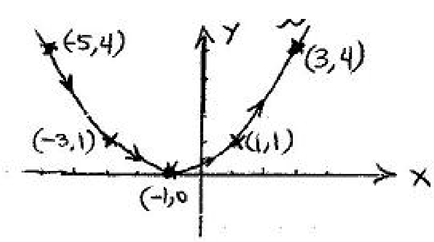
Step 3- $l: \begin{cases} x(t) = 1 + 6t \\ y(t) = -2 - 10t \\ z(t) = 10 + 8t \end{cases}$ if you use $\vec{r}\,'(t)$ or $l: \begin{cases} x(t) = 1 + \dfrac{3\sqrt{2}}{10}t \\ y(t) = -2 - \dfrac{\sqrt{2}}{2}t \\ z(t) = 10 + \dfrac{2\sqrt{2}}{5}t \end{cases}$ if you use $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$

Graph of parametric equations in 2D Space

Method 1: Evaluate the vector function at different values of t (time).

Example: $\begin{cases} x(t) = 2t - 1 \\ y(t) = t^2 \end{cases}$ Plot x and y only and indicate the direction of increasing time

t	x	y
-2	-5	4
-1	-3	1
0	-1	0
1	1	1
2	3	4



Method 2: Eliminate the parameter "t" and write the equation in terms of x and y only

Example: $\begin{cases} x(t) = (v_0 Cos\theta)t \\ y(t) = \frac{-1}{2}gt^2 + (v_0 Sin\theta)t \end{cases}$ solve for $t = \frac{x(t)}{v_0 Cos\theta}$ plug it into $y(t)$

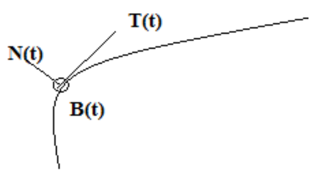
$$y(t) = \frac{-1}{2}g(\frac{x(t)}{v_0 Cos\theta})^2 + (v_0 Sin\theta)(\frac{x(t)}{v_0 Cos\theta}) = (\frac{-g}{2v^2{}_0 Cos^2\theta})x^2 + (Tan\theta)x$$

This quadratic equation with a negative coefficient of x^2 is a concave down parabola and guarantees object with mass and low velocity will return back to earth. Objects with high velocity more than 7 mi/sec can escape the earth gravitational field. (A bit over 30 times the speed of sound). How do you explain the fact that high velocity escapes the earth with the above equation?

Unit Normal vector and Binormal vector

Unit normal vector is the unit vector of rate of change of unit tangent vector $\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$.

Binormal vector is the cross product of unit tangent vector with unit normal vector $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$ therefor $\vec{B}(t) \perp \vec{T}(t)$ *and* $\vec{B}(t) \perp \vec{N}(t)$



Bi-normal is coming out of the page and it is orthogonal to the plane created by tangent and normal vectors

Example: $\vec{r}(t) =< t,\ t^2 > \quad \Rightarrow \quad \vec{r}'(t) =< 1,\ 2t > \quad \Rightarrow \quad \vec{r}''(t) =< 0,\ 2 >$

$\vec{r}(t),\ \vec{r}'(t),\ and\ \vec{r}''(t)$ are not orthogonal to each other except at $t = 0$.

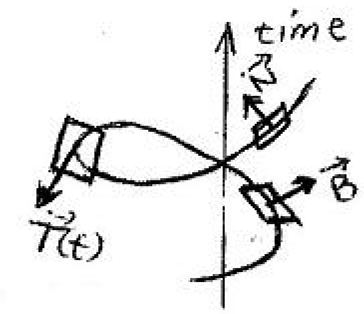
Let's look at the unit tangent and normal vectors.

$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|} = \frac{<1,\ 2t>}{\sqrt{4t^2+1}}$ $\quad \vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \frac{<-2t,\ 1>}{\sqrt{4t^2+1}}$ these two are orthogonal to each other, So $\vec{T}(t) \bullet \vec{N}(t) = 0$

$$\vec{B}(t) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2t & 0 \\ -2t & 1 & 0 \end{vmatrix} \frac{1}{4t^2+1} = \frac{1}{4t^2+1}(4t^2+1)\hat{k} =< 0,\ 0,\ 1 >$$

This shows that Bi-normal is independent of time and it is always constant, therefore the curve is in 2D.

Tangent plane, Normal plane, and osculating plane



Plane is a point and its Normal vector

- Tangent plane is a point and Normal vector
- Normal plane is a point and Tangent plane
- Osculating plane is a point and Bi-normal plane

Homework set 2 (6 questions)

Problem: Find the tangent line to a parametric equation at the given point.

1. $x(t) = 4\sqrt{t}, \quad y(t) = t^2 - 10, \quad z(t) = 4/t, \quad P(8,\ 6,\ 1)$

2. $\vec{r}(t) =< t^2, tSin(t), tCos(t) > \qquad t = 1$

3. $\vec{r}(t) =< tSin(t),\ tCos(t),\ t > \qquad P(\pi/2,\ 0,\ \pi/2)$

4. $\vec{r}(t) =< 10t,\ -5t^2 + 10t >$ Find $\vec{r}\,'(t),\ \vec{r}\,''(t),\ and\ \vec{T}(t)$ at the following times 0, 1, and 2s.

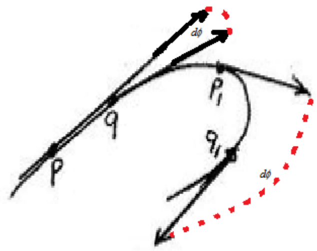
5. $\vec{r}(t) =< 10t,\ -5t^2 + 10t >$ Graph $\vec{r}\,'(t),\ \vec{r}\,''(t)$ and find the distance traveled (Arc length).

6. $\vec{r}(t) =< t,\ t^2 >$ Write an equation for Tangent plane, Normal plane, and osculating plane at $t = 0,\ t = 1\sec$

Curvature

Curvature is defined as rate of change of unit tangent vector per unit length $\kappa = \left|\frac{d\vec{T}}{d\vec{s}}\right|$.

We can use a chain rule to write curvature in terms of time $\kappa = \left|\frac{d\vec{T}}{d\vec{s}}\right| = \left|\frac{d\vec{T}/dt}{d\vec{s}/dt}\right| = \left|\frac{\vec{T}'(t)}{\vec{r}'(t)}\right|$



Space curvature creates acceleration

Look at arc pq and the angle difference between tangential vectors at end points. Look at the arc p_1q_1 [which is the same length as arc pq] and the angle difference between tangential vectors at end points. Larger Curvature has larger angle difference in Tangent vectors and smaller the radius of curvature. ($\kappa = \frac{1}{\rho}$)

Acceleration: Start with $\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$ then we can rearrange it to $\vec{r}'(t) = |\vec{r}'(t)|\ \vec{T}(t)$.

Differentiate to get $\vec{r}''(t) = \left(\frac{d^2s}{dt^2}\right)\vec{T}(t) + \left(\frac{ds}{dt}\right)\ \vec{T}'(t)$

The 2nd term was expanded by chain rule as follow $= \left(\frac{d^2s}{dt^2}\right)\vec{T}(t) + \left(\frac{ds}{dt}\right)\left(\frac{dT}{ds}\right)\left(\frac{ds}{dt}\right)$

$\left(\frac{ds}{dt}\right)\left(\frac{dT}{ds}\right)\left(\frac{ds}{dt}\right) = (\frac{ds}{dt})^2\left(\frac{dT}{ds}\right) = v^2\left(\frac{dT}{ds}\right)$,

$\left(\frac{dT}{ds}\right) = \kappa\ \vec{N} \qquad v^2\kappa = \frac{v^2}{\rho} = a_N \qquad \left(\frac{d^2s}{dt^2}\right) = a_T$

So $\vec{r}''(t) = a_T\vec{T} + a_N\vec{N}$ Equation of acceleration involves curvature

Now we can intrudes a new set of Bases $\{\vec{T}(t),\ \vec{N}(t),\ \vec{B}(t)\}$. These set of Bases are function of time and they are not fixed in space time unlike $\{\hat{i},\ \hat{j},\ \hat{k}\}$.

Let's find curvature

We start with this product

$$\vec{r}'\times\vec{r}''=\begin{vmatrix}\vec{T} & \vec{N} & \vec{B}\\ \frac{ds}{dt} & 0 & 0\\ \frac{d^2s}{dt^2} & \frac{ds}{dt}\left|\vec{T}'\right| & 0\end{vmatrix}=\vec{B}(\frac{ds}{dt})^2\left|\vec{T}'\right|=(\vec{T}\times\vec{N})(\frac{ds}{dt})^2\left|\vec{T}'\right|=(\vec{T}\times\vec{T}')(\frac{ds}{dt})^2$$ Since $\vec{N}\left|\vec{T}'\right|=\vec{T}'$

Since $\left|\vec{T}\right|=1$ then $\left|\vec{r}'\times\vec{r}''\right|=\left|\vec{T}'\right|(\frac{ds}{dt})^2$ and then $\left|\vec{T}'\right|=\dfrac{\left|\vec{r}'\times\vec{r}''\right|}{(\frac{ds}{dt})^2}$ and $\kappa=\left|\dfrac{\vec{T}'(t)}{\vec{r}'(t)}\right|=\dfrac{\left|\vec{r}'\times\vec{r}''\right|}{\left|\vec{r}'(t)\right|^3}$

Practice: How does the formula for Curvature look like in other coordinate systems?

In rectangular coordinates $\kappa=\dfrac{\left|-f''(x)\right|}{(1+[f'(x)]^2)^{3/2}}$ Hint: Use $\vec{r}(x)=<x, f(x)>$

In Polar coordinates $\kappa=\dfrac{\left|2(r')^2-rr'+r^2\right|}{(r'^2+r^2)^{3/2}}$ Hint: Use $\vec{r}(x)=<rCos\theta, rSin\theta>$

Theorem: if $\vec{r}(t)$ is differentiable and $\left|\vec{r}(t)\right|=const$ then $\vec{r}'(t)\bullet\vec{r}(t)=0$

If $\left|\vec{r}(t)\right|=const$, $\vec{r}(t)\bullet\vec{r}(t)=\vec{r}^2(t)=const$, $\frac{d}{dt}(\vec{r}(t)\bullet\vec{r}(t))=2\vec{r}(t)\bullet\vec{r}'(t)=\frac{d}{dt}Const=0$

Homework Set 3 (5 questions)

1- Derive the formula for Curvature in

a) Rectangular coordinates $\kappa=\dfrac{\left|-f''(x)\right|}{(1+[f'(x)]^2)^{3/2}}$

b) Polar coordinates $\kappa=\dfrac{\left|2(r')^2-rr'+r^2\right|}{(r'^2+r^2)^{3/2}}$

c) Parametric coordinates $\kappa=\dfrac{\left|f'g''-g'f''\right|}{[f'^2+g'^2]^{3/2}}$

2- Given $\vec{r}(t)=<t^2, t>$

a) Graph $\vec{r}(t)$

b) Find $\vec{T}(1),\ \vec{N}(1)$ Then Draw them

c) Find the arc-length of the curve $[0,1]$

d) Find $\kappa(0),\ \kappa(1).$

3- Given $\vec{r}(t) = < rCos(\omega t), rSin(\omega t) >$ Show that $\vec{r}(t).\vec{v}(t) = 0$, $\vec{a}(t).\vec{v}(t) = 0$, and $\vec{a} = -\omega^2 \vec{r}$

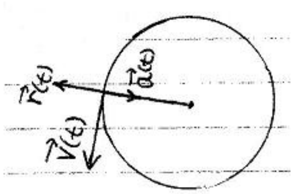


Figure for problem 3

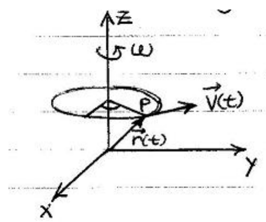


Figure for problem 4

4- A point is rotating about z axis on a circle of radius r that lies on the plane $z = h$ with a constant angular speed $\omega = d\theta / dt$. Show that $\vec{\omega}(t) \times \vec{r}(t) = \vec{v}(t)$.

5- Use Newton's 2nd law $\sum_{all} \vec{F} = m\vec{a}$ for a constant mass in addition to the information from the last problem $\vec{\omega}(t) \times \vec{r}(t) = \vec{v}(t)$ and combined with $\vec{a} = -\omega^2 \vec{r}$ from problem 3, show that $\sum_{all} \vec{F} = m\dfrac{v^2}{r}$

Acceleration is $\vec{a} = a_T \vec{T} + a_N \vec{N} = v'\vec{T} + \kappa v^2 \vec{N}$ where $\kappa = 1/\rho$. And ρ is radius of Curvature.

$$a_T = v' = Comp_{\vec{v}}^{\vec{a}} = \frac{\vec{v} \bullet \vec{a}}{|\vec{v}|} = \frac{\vec{r}'(t) \bullet \vec{r}''(t)}{|\vec{r}'(t)|}$$

$$a_N = \kappa v^2 = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'(t)|^3} |\vec{r}'(t)|^2 = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|}$$

Some basic parametrized curved

$$Circle \begin{cases} x(t) = h + rCos(\omega t) \\ y(t) = k + rSin(\omega t) \end{cases} \qquad Ellipse \begin{cases} x(t) = h + aCos(\omega t) \\ y(t) = k + bSin(\omega t) \end{cases}$$

$$Parabola \begin{cases} x(t) = v_x t + x_0 \\ y(t) = \frac{1}{2} a t^2 + v_y t + y_0 \end{cases} \qquad Line \begin{cases} x(t) = v_x t + x_0 \\ y(t) = v_y t + y_0 \end{cases}$$

$$Hyperbola \begin{cases} x(t) = h + aCosh(\omega t) \\ y(t) = k + bSinh(\omega t) \end{cases} \qquad Hyperbola \begin{cases} x(t) = h + aSec(\omega t) \\ y(t) = k + bTan(\omega t) \end{cases}$$

Re-Parametrize the curve respect to s

Step1: Given $\vec{r}(t)$, find $|\vec{r}\,'(t)|$

Step2: Evaluate $s = \int_0^t |\vec{r}\,'(t^*)| dt^*$ and then solve for t in terms of s.

Step3: Replace t by function of s in $\vec{r}(t)$.

a) Given $\vec{r}(t) =< \frac{2}{1+t^2} - 1, \ \frac{2t}{1+t^2} >$ and $\vec{r}\,'(t) =< \frac{-4t}{(1+t^2)^2}, \ \frac{2-2t^2}{(1+t^2)^2} >$ therefor $|\vec{r}\,'(t)| = \frac{2}{1+t^2}$

b) $s = \int_0^t \frac{2}{1+t^{*2}} dt^* = 2Tan^{-1}(t) \qquad t = Tan(s/2)$

c) Replace t by function of s in $\vec{r}(t)$. Use trig identities to get $\vec{r}(s) =< Cos(s), \ Sin(s) >$

Homework Set 4 (7 questions)

1- Show the work for Re-parametrize $\vec{r}(t) =< \frac{2}{1+t^2} - 1, \ \frac{2t}{1+t^2} >$ into $\vec{r}(s) =< Cos(s), \ Sin(s) >$

2- For curve $\vec{r}(t) =< Sin(t), \ t, \ Cos(t) >$ Show that

$a) \ |\kappa| = |\vec{r}\,''(s)| \qquad b) \ \vec{r}\,'(s) \bullet \vec{r}\,''(s) = 0 \qquad c) \ \vec{r}\,'(s) \bullet \vec{r}\,'(s) = 1$

3- Find the curvature of the following polar curves

$a) \ r = 1 + Sin\theta \qquad b) \ r = aSin\theta \qquad c) \ r = \theta \qquad d) \ r = e^{\theta}$

4- Find $a_T \ and \ a_N$ for the curve given by $\vec{r}(t) =< 2t, \ t^2, \ (-1/3)t^3 >$.

5- Find the following differentiations

$a)\ \frac{d}{dt}[\vec{r}(t)\times\vec{r}\,'(t)]=$ $b)\ \frac{d}{dt}[\vec{u}\bullet(\vec{v}\times\vec{w})]=$

6- Let $\vec{r}(t)=<x(t),\ \ y(t),\ \ z(t)>$ Show that $\frac{d}{dt}\vec{r}(t)=\frac{\vec{r}(t)\bullet\vec{r}\,'(t)}{|\vec{r}(t)|}$

7- If $u(t)=\vec{r}(t)\bullet[\vec{r}\,'(t)\times\vec{r}\,''(t)]$ show that $u'(t)=\vec{r}(t)\bullet[\vec{r}\,'(t)\times\vec{r}\,'''(t)]$

Show that Exercises

1-Show that $\frac{d\vec{T}}{ds}=\kappa\ \vec{N}$

$$\frac{d\vec{T}}{ds}=\frac{\frac{d\vec{T}}{dt}}{\frac{ds}{dt}}\left(\frac{\left|\frac{d\vec{T}}{dt}\right|}{\left|\frac{d\vec{T}}{dt}\right|}\right)=\frac{\left|\frac{d\vec{T}}{dt}\right|}{\frac{ds}{dt}}\ \frac{\frac{d\vec{T}}{dt}}{\left|\frac{d\vec{T}}{dt}\right|}=\kappa\ \vec{N}$$

2- Show that $\kappa=\left|\frac{d\phi}{ds}\right|$ $\qquad$ $Given\ \vec{T}=T<Cos\phi,Sin\phi>$

$$\kappa=\left|\frac{d\vec{T}}{ds}\right|=\left|\frac{d\vec{T}}{d\phi}\frac{d\phi}{ds}\right|=\left|\frac{d\vec{T}}{d\phi}\right|\left|\frac{d\phi}{ds}\right|=\left|<-Sin\phi,\ \ Cos\phi>\right|\left|\frac{d\phi}{ds}\right|=\left|\frac{d\phi}{ds}\right|$$

If $\left|\frac{d\vec{T}}{ds}\right|=\left|\frac{d\phi}{ds}\right|$ does not mean that $\vec{T}=\phi$

3- Show that $\frac{d\vec{B}}{ds}\perp\vec{B}$

$$\frac{d\vec{B}}{ds}\bullet\vec{B}=\frac{d}{ds}(\vec{T}\times\vec{N})\bullet\vec{B}=(\frac{d}{ds}\vec{T}\times\vec{N})\bullet\vec{B}+(\vec{T}\times\frac{d}{ds}\vec{N})\bullet\vec{B}=(\kappa\vec{N}\times\vec{N})\bullet\vec{B}+(\vec{T}\times(a\vec{T}+b\vec{B}))\bullet\vec{B}=0$$

4- Show that $\frac{d\vec{B}}{ds}\perp\vec{T}$

$$\frac{d\vec{B}}{ds}\bullet\vec{T}=\frac{d}{ds}(\vec{T}\times\vec{N}).\vec{T}=\frac{1}{|\vec{r}\,'|}\frac{d}{dt}(\vec{T}\times\vec{N}).\vec{T}=\frac{1}{|\vec{r}\,'|}[(\vec{T}\,'\times\vec{N})+(\vec{T}\times\vec{N}\,')].\vec{T}$$

$=\frac{1}{|\vec{r}\,'|}[\vec{T}\bullet(\vec{T}\,'\times\vec{N})+\vec{T}\bullet(\vec{T}\times\vec{N}\,')]=0$ Since $\vec{T}\,'\times\vec{N}=0$ and $\vec{T}\bullet(\vec{T}\times\vec{N})=0$ then $\frac{d\vec{B}}{ds}\bullet\vec{T}=0$

5- Show that $\frac{d\vec{B}}{ds} = -\tau_{(s)}\vec{N}$ Where $\tau_{(s)}$ is torsion of the curve

$\vec{N} = \vec{B} \times \vec{T}$ Since $\frac{d\vec{B}}{ds} \perp \vec{T}$ and $\frac{d\vec{B}}{ds} \perp \vec{B}$ then $\frac{d\vec{B}}{ds} // \vec{N}$ which means $\frac{d\vec{B}}{ds} = a\vec{N}$ Where $a = \tau_{(s)}$

6- Show that for a plane carve $\tau_{(s)} = 0$

For a plane curve, unit Tangent vector and unit Normal vectors stay in a plane then Binormal vector stay constant so the rate of change Bi-normal in respect to arc-length is zero $\frac{d\vec{B}}{ds} = 0$ therefor the torsion is zero.

7- Show that $\frac{d\vec{N}}{ds} = \tau_{(s)}\vec{B} - \kappa\vec{T}$

$$\frac{d\vec{N}}{ds} = \frac{d}{ds}(\vec{B} \times \vec{T}) = (\frac{d}{ds}\vec{B} \times \vec{T}) + (\vec{B} \times \frac{d}{ds}\vec{T}) = (-\tau_{(s)}\vec{N} \times \vec{T}) + (\vec{B} \times \kappa\ \vec{N}) = \tau_{(s)}\vec{B} - \kappa\vec{T}$$

Vectorial Mechanics

In classical Mechanics the momentum $\vec{P}$ of an object is defined by $\vec{P}(t) = m\vec{v}(t)$ (a scalar multiplication). The net force is the rate of change of momentum $\vec{F} = \frac{d\vec{P}}{dt} = \frac{d}{dt}(m\vec{v})$ $= \frac{dm}{dt}\vec{v} + m\frac{d}{dt}(\vec{v})$ for a non-changing mass the net force is product of mass and acceleration. $\vec{F} = m\frac{d}{dt}(\vec{v})$.

Work is dot product of force and displacement $W = \int_a^b \vec{F} \bullet d\vec{s}$ for non-changing force the work is simply $W = \vec{F} \bullet (\vec{s})$

The power is rate of change of work per unit time $P = \frac{dW}{dt} = \frac{d}{dt}(\vec{F} \bullet \vec{s})$ for non-changing force $P = \frac{dW}{dt} = \vec{F} \bullet \frac{d}{dt}(\vec{s}) = \vec{F} \bullet \vec{v}$

Kinetic Energy $KE = \frac{1}{2}m(\vec{v} \bullet \vec{v}) = \frac{1}{2}mv^2$

Dynamic of rotations

Torque is cross product of moment arm and applied force $\vec{\Gamma}(t) = \vec{r}(t) \times \vec{F}(t)$

Angular momentum is cross product of moment arm and linear momentum $\vec{L}(t) = \vec{r}(t) \times \vec{P}(t)$

So $\frac{d}{dt}\vec{L}(t) = \vec{r}(t) \times \vec{F}(t) = \vec{\Gamma}$ Torque is equal to rate of change of angular momentum.

Central forces, forces which act along the line of the position vector, produce no torque.

Given $\vec{F}(t) = C(t)\vec{r}(t)$ where $C(t)$ is a scalar function

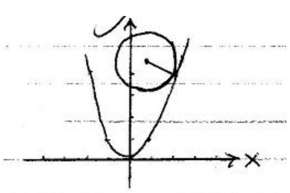
$\vec{\Gamma}(t) = \vec{r}(t) \times \vec{F}(t) = \vec{r}(t) \times C(t)\vec{r}(t) = 0$

If angular momentum is constant then $\vec{\Gamma}(t) = \frac{d\vec{L}}{dt} = 0$, there will be no torque generated.

Homework set 5 (6 questions)

1- Show that for central forces, angular momentum is constant and also if the angular momentum is constant, the force is zero or it is a central force.

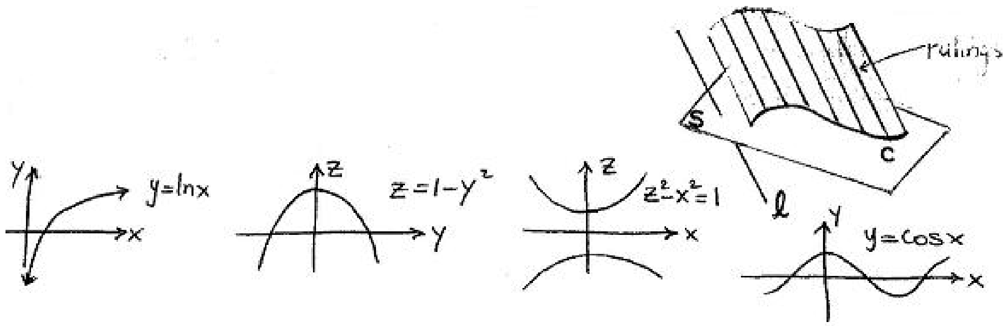
2- Find curvature for $y = x^2$ at $(0,0),\ (1,1),\ and\ (2,4)$. Find radius of curvature for each and then find equation of circle tangent to the curve at point $(2,4)$.



3- Find equation of circle tangent to curve $y = x^2$ at point x. Then find the rate of change of area of the circle per length. Is the rate of change concave up or down? Is the rate of change increasing or decreasing?

What is the graph of $x^2 + y^2 = a^2$ in 3-D? If C is a curve in plane S and L is a line not parallel to S, then the set of all points generated by moving line traversing C parallel to L is called Cylinder and the curve C is called the directrix of the cylinder.

4- Graph the following in 3D.



A sphere is the set of all points P in 3D –Space that are equidistant from a fixed point called the center. $x^2+y^2+z^2=\rho^2$

Quadratic Surfaces: $Ax^2+By^2+Cz^2+Dx+Ey+Fz+G=0$ with no cross terms.

a) Ellipsoid: $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

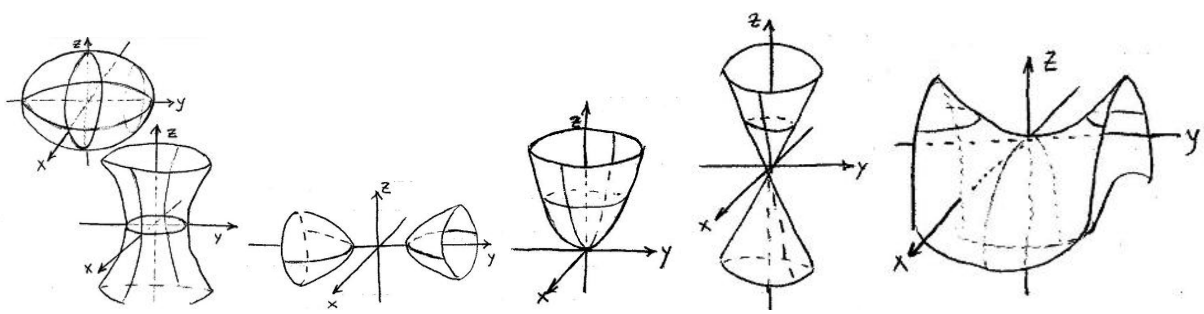
b) Hyperboloid of one sheet $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$

c) Hyperboloid of two sheets $\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=-1$

d) Paraboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}=cz$

e) Cone $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}$

f) Hyperbolic paraboloid $\frac{x^2}{a^2}-\frac{y^2}{b^2}=cz$



5- Cut each of the above quadratic surfaces with three planes $x=0,\ \ y=0$ and $z=0$. What is the name of the curve of intersection of the planes and quadratic surfaces?

Surface of revolutions: If there is any axis of symmetry

Equation of the Curve C	*Axis of symmetry*	*Equation of Surface S*
$f(x,y)=0$	$x-axis$	$f(x,\ \pm\sqrt{y^2+z^2})=0$
$f(x,y)=0$	$y-axis$	$f(\pm\sqrt{x^2+z^2},\ \ y)=0$

6- Write out the Equation of Surface S for the following Curves rotated around an axis:

$a)$ $y = x^2$ Rotated around $y-axis$

$b)$ $4x^2 + y^2 = 16$ Rotated around $x-axis$

$c)$ $y = z$ Rotated around $z-axis$