TABLE CONTENT

Review from Pre-Calculus Vector in 3-D

The most common tools in Physics (Sciences) are vector notations. They are abstract tools that can help us to visualize imaginary fields such as Electric or Magnetic fields. To make Vectors more tangible, we give them Magnitude (Length) and Direction (Angle)

Given a vector from starting point $P(P_x, P_y, P_z)$ to terminal point $Q(Q_x, Q_y, Q_z)$,

$\vec{PQ} = < Q_x - P_x, Q_y - P_y, Q_z - P_z >$ with length of $D_{PQ} = \sqrt{(Q_x - P_x)^2 + (Q_y - P_y)^2 + (Q_z - P_z)^2}$

angles of $\alpha = Tan^{-1}\dfrac{\sqrt{y^2 + z^2}}{x}$, $\beta = Tan^{-1}\dfrac{\sqrt{x^2 + z^2}}{y}$, $\phi = Tan^{-1}\dfrac{\sqrt{x^2 + y^2}}{z}$ with x, y, z-axis.

In 2-Dimension, when $z = 0,\ then\ \phi = 90^\circ,\ \alpha = Tan^{-1}\dfrac{|y|}{x},\ \beta = Tan^{-1}\dfrac{|x|}{y}$. You can show that

$Cos^2\alpha + Cos^2\beta + Cos^2\phi = 1$ (Sum of squares of directional cosines is one)

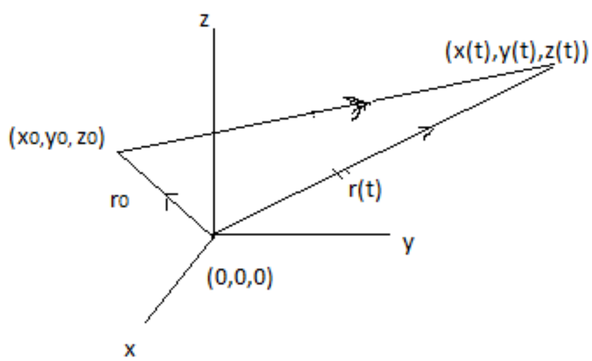
Vector $\vec{V} = < A, B, C >$ has a magnitude of $|V| = \sqrt{A^2 + B^2 + C^2}$ with unit vector of $\vec{U}_v = \dfrac{\vec{V}}{|V|}$

Equation of Sphere: Location of all the points that are equidistance to a point [which call center (x_0, y_0, z_0)] is surface of Spheres. Its equation is

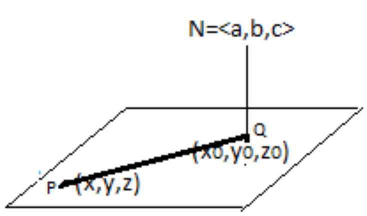
$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = R^2$(Where R is the distance of all the points to the center). Use Pythagorean to show distance formula and then use it to prove the formula for equation of Sphere.

Equation of a line: A line is defined by a point and a direction. In parametric form with a point P (x_1, y_1, z_1) and directional vector $\vec{V} = < a, b, c >$ is

$x = x_1 + at$, $y = y_1 + bt$, $z = z_1 + ct$ where the parameterized variable t is Time.



Equation of a plane: A plane is defined by a point and a normal vector (A vector which is perpendicular to the plane). The equation of a plane with a point P (x_1, y_1, z_1) and normal vector $\vec{N} = < a, b, c >$ is $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ *or* $ax + by + cz = d$



Given a point $Q(x_0, y_0, z_0)$ on plane S and a Normal vector $\vec{N} = < a, b, c >$ to the plane S, for any point $P(x, y, z)$ on plane S, the vector $\vec{PQ} = < x - x_0, y - y_0, z - z_0 >$ is orthogonal to $\vec{N} = < a, b, c >$. $\vec{PQ} \bullet \vec{N} = 0$ or $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$ *or* $ax + by + cz = d$

Dot product /inner product/ scalar product is defined as $\vec{A} \bullet \vec{B}$. (Two ways to find it)

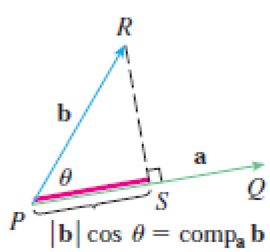
If the components are given then $\vec{A} \bullet \vec{B} = a_1 b_1 + a_2 b_2 + a_3 b_3$

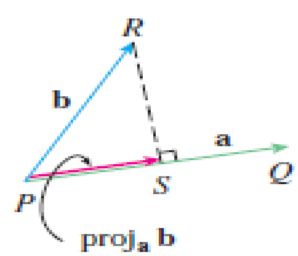
If the angle between the vectors is given $\vec{A} \bullet \vec{B} = |A||B|\cos\theta$

Use Dot product to show the following

a) The angle between two vectors if the components are given $\theta = Cos^{-1} \dfrac{a_1 b_1 + a_2 b_2 + a_3 b_3}{|A||B|}$

b) Component of $\vec{A}$ *on* $\vec{B}$ = $Comp_B^A = \dfrac{\vec{A} \bullet \vec{B}}{B}$, Projection of $\vec{A}$ *on* $\vec{B}$ = $\Pr oj_B^A = (\dfrac{\vec{A} \bullet \vec{B}}{B})\vec{U}_B$





c) The angle between two lines. (Use their directional vectors)

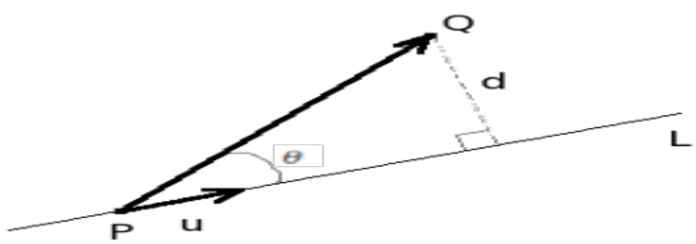
d) The angle between two planes (Use their Normal vectors)

e) The angle between a line and a plane (use Normal vector of the plane and directional vector of the line)

f) Distance of point Q (x_2, y_2, z_2) to line $\begin{cases} x = x_1 + at \\ y = y_1 + bt \\ z = z_1 + ct \end{cases}$

Step1: Find vector PQ **Step2:** Find Component of PQ on directional vector V

Step3: (Use Pythagorean) distance of the point to the line $D = \sqrt{(|PQ|)^2 - (Comp_V^{PQ})^2}$



g) Distance of point Q (x_2, y_2, z_2) to Plane $ax + by + cz = d$

Step1: Find vector PQ **Step2:** Find Com of PQ on Normal vector N D= $Comp_N^{PQ}$

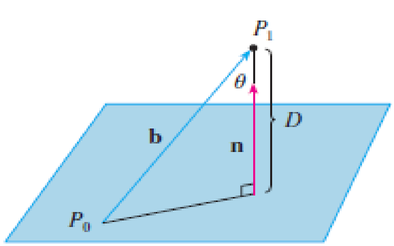


FIGURE 12

h) Distance of a line to a parallel plane

Step1: Choose randomly a point P on the line

Step2: Find the distance from P to the plane

i) Point of intersection of $x = x_1 + at$, $y = y_1 + bt$, $z = z_1 + ct$ with $ax + by + cz = d$

Step1: Sub. for variables **Step2:** Solve for t (time)

Step3: Find the coordinate of the point.

j) Line of intersection of $ax + by + cz = d$ with $a_1x + b_1y + c_1z = d_1$

Step1: Call one variable t **Step2:** Solve for the system for other two variables

Step3: Now you have x,y, and z all in terms of t. That is an equation of line of intersection.

Cross product /outer product/ vector product is defined as $\vec{A} \times \vec{B}$ in two ways

a) With the components $\left|\vec{A} \times \vec{B}\right| = \begin{vmatrix} i & j & k \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \langle a_y b_z - a_z b_y, a_z b_x - a_x b_z, a_x b_y - a_y b_x \rangle$

b) With the angle between the vectors $\left|\vec{A} \times \vec{B}\right| = |A||B| Sin\theta$ (Direction is right hand rule)

Use Cross product to show the following

a) The angle between two vectors if the components are given $\theta = Sin^{-1} \dfrac{\left|\vec{A} \times \vec{B}\right|}{|A||B|}$

b) The relation between Cross product and Dot product $\left|\vec{A}\times\vec{B}\right| = \sqrt{(|A||B|)^2 - (\vec{A}\bullet\vec{B})^2}$

c) Equation of a plain with three given points A,B, and O

Step1: Find $\vec{OA}$ and $\vec{OB}$ **Step2:** Find $\vec{N} = \vec{OA}\times\vec{OB} = < a, b, c >$

Step3: Use any of the points and normal vector $a(x-x_1)+b(y-y_1)+c(z-z_1)=0$

d) Equation of a plain with point Q (x_2, y_2, z_2) and line $\begin{cases} x = x_1 + at \\ y = y_1 + bt \\ z = z_1 + ct \end{cases}$

Step1: Find vector PQ **Step2:** Find Cross product of PQ and directional vector V

Step3: Use point Q and normal vector $a(x-x_2)+b(y-y_2)+c(z-z_2)=0$

e) Area of a triangle with three given points A,B, and O Area = $\frac{1}{2}\left|\vec{OA}\times\vec{OB}\right|$

f) Volume of a parallelepiped with three vectors $\vec{A},\vec{B},\vec{C}$

Step1: Find $\vec{D} = \vec{A}\times\vec{B}$ **Step2:** Find (Component of $\vec{C}$ on $\vec{D}) = Comp_{\vec{D}}^{\vec{C}}$ =Height

Step 3: Volume = Area X height V= $\left|\vec{A}\times\vec{B}\right|$ X $Comp_{\vec{D}}^{\vec{C}}$ = $\vec{C}\bullet\vec{D}$

Graph of Line, Plane, Sphere

Line: Plot the given point and stretch it in the direction of its directional vector. Or pick two different values for time T and evaluate the x, y, z. Then ploy the points and connect them.

Plane: Find x, y, and z intercepts and connect them. If any of variables is missing in the equation then draw a line with two other variables and elongated in the direction of missing variable. If two variables are missing then the plane is elongated in two directions.

Sphere: Plot the center and stretch the center with length R along positive and negative of x, y, and z axes. Then connect the end points with circular curves.

Chapter 11 vectors

We all started learning mathematics with arithmetic (Numbers and Basic operations).

You have been misled to memorize order of operations blindly and use it over and over until you thought that you had the mastery of these basic operations (Since there were so many rules). All is well since you had the mastery of these basic operations but as you moved up to Algebra and pre-Calculus, you came across more operations and more rules.

Let's learn mathematics with less rules and a better conceptual understanding.

All we need addition and scalar multiplication as two operations to do all arithmetic and vector algebra.

Numbers are scalar with no directions such as mass, Energy, distance, and speed.

You know how to solve problems such as pulling a chain with μ which a mass M is attached to the end of it. Easily you use concept of super-position to add the work needed to pull the chain and then added to the work needed to pull the mass. Now if there was multiple identical masses were attached, then simply you multiply the energy needed for one mass to pull up and then multiply that value by the number of masses. This is called scalar multiplication.

How do the mathematics of vectors which possess the magnitude and direction work?

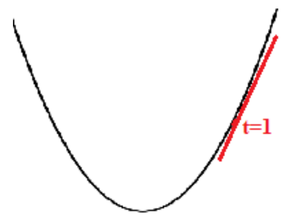
One can define a direction in a few ways.

1- Introduce the starting and terminal point of the vector.
2- Give the length and angles with respect to some known axis.
3- Describe the components of the vector.

Some of the vectors which you have seen in the past including displacement, Velocity and acceleration are mathematical properties of an object. The vector fields such as Force Field, Electric, and Magnetic Fields are assumed to be measured at a point and they are properties of the space only and we assume that they don't change in time.

Each vector can be resolve in to its components; number of non-zero components gives you the rank of the vector (The minimum dimension needed for the vector to be in that space). If $\vec{V} = < 3,4,0 >$ then the rank of V is 2 in a 3D space. This vector is fixed in time since the magnitude of it is in dependent of time. So, it can be a vector in a vector field which was measured at a point in time or space. For the vectors which are time dependent, then we can differentiate and integrate them respect to time.

Example1: A particle with position vector $\vec{r}(t) = <3t+2,\ t^2+2t-10>$ moves on a curve.



Find the following for this particle at time $t = 1\sec$.

a) Velocity $\vec{v} = \frac{d\vec{r}}{dt} = <3,\ 2t+2>\ \ At\ t = 1\sec\ \ \vec{r}' = <3,\ 2(1)+2> = <3,4>$ Speed $= |\vec{v}| = 5m/s$

b) Acceleration in Tangential direction $a_T = <0,\ 2>$ at any time $a_T = 2m/s^2$

c) Force in Tangential direction if the particle mass is 4kg. $F_T = ma_T = 4kg2m/s^2 = 8N$

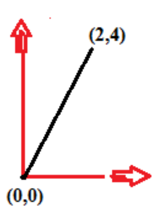
Example 2: A force $\vec{F} = <3t^2,\ 2t-1>N$ is excreted to a particle in duration $0 \le t \le 2\sec$.

What is its momentum?

Momentum we know by Newton's 2nd law that $\vec{F} = \frac{d\vec{p}}{dt}$ so $P = \int F.dt = \int_0^2 <3t^2,\ 2t-1>dt$

$$P = <\int_0^2 3t^2 dt,\ \int_0^2 (2t-1)dt> = <t^3+c_1,\ t^2-t+c_2>\ \ 0 \le t \le 2\ \ \vec{P} = <8\ 2>N/Sec$$

Example3: A particle under the influence of a force field $\vec{F}(x) = <2x, 4y>N$ moves from origin to point (2, 4). Find the work done. Note that the force is a function of space and it contains the mass of the particle.



$$W = Sum\ \ <2x,4y>.<dx,dy> = \int_0^4 \int_0^2 2xdx + 4ydy = \int_0^4 4ydy + \int_0^2 2xdx = 2y^2\Big|_0^4 + x^2\Big|_0^2 = 32 + 4 = 36 Joule$$

Practice1: A particle with position vector $\vec{r}(t) =< RCos(\omega t),\ \ RSin(\omega t) >$ moves on a curve.

Find the following for this particle.

a) Graph of the curve in 2D

b) Velocity vector $\vec{V}$

c) Show that the velocity vector is orthogonal to $\vec{r}(t)$

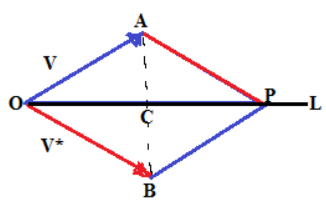
d) Speed $\left|\vec{V}\right| = v$

b) Acceleration and then show that it is anti-parallel to $\vec{r}(t)$

c) Show that the force in normal direction is $F_{\perp} = M\dfrac{v^2}{R}$ for any mass M

d) Find its Momentum

Example4: Given vector $\vec{V}$ and line L with directional vector $\vec{U}$. Find the reflection of vector V respect to line L.



$OC = PROJ_{\vec{u}}^{\vec{v}}$ and $OP = 2PROJ_{\vec{u}}^{\vec{v}}$ we know $\vec{V} + \vec{V}^* = 2PROJ_{\vec{u}}^{\vec{v}}$ Then $\vec{V}^* = 2PROJ_{\vec{u}}^{\vec{v}} - \vec{V}$

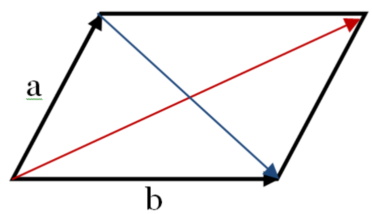
Practice2: Find reflection of vector $\vec{V} =< 3,\ 1 >$ on line $L: x = 2 - t\ \ y = 1 - 3t$.

Some Geometry by Vector Methods

Proposition 1: The diagonals of a parallelogram are perpendicular iff the parallelogram is a rhombus.

Since the diagonals of a parallelogram are perpendicular, we have:

$(a+b)\bullet(a-b)=0$

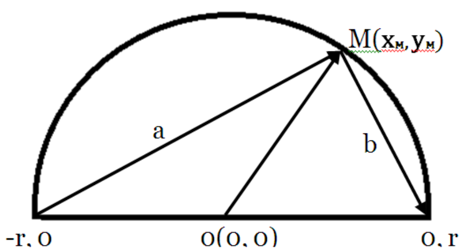


$aa-ab+ba-bb=0 \qquad |a|^2=|b|^2$

➔ $|a|=|b|$

➔ Parallelogram is a rhombus

Proposition 2: Every angle inscribed in a semicircle is a right angle.



$a=<x_m+r, y_m> \qquad b=<r-x_m, -y_m> \qquad a\bullet b=(x_m+r)(r-x_{mm})-y_m^2=r^2-(x_m^2+y_m^2)$ **(1)**

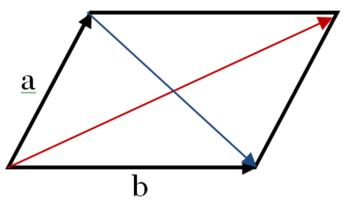
$\overrightarrow{OM}=<x_m, y_m> \qquad |\overrightarrow{OM}|^2=r^2=x_m^2+y_m^2$ **(2)**

From **(1)** and **(2)**, we have **a . b** $= r^2-r^2=0$

➔ **a** is perpendicular to **b**

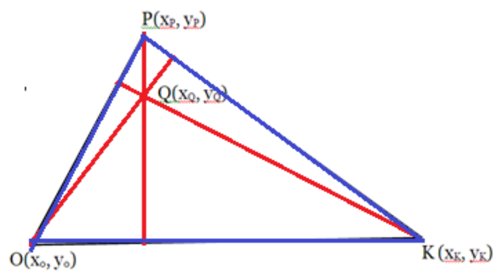
➔ Every angle inscribed in a semicircle is a right angle.

Proposition 3: In a parallelogram the sum of squares of the lengths of the diagonals equals the sum of the squares of the lengths of the sides.



$|a+b|^2 + |a-b|^2 = |a|^2 + |b|^2 + 2|a||b| + |a|^2 + |b|^2 - 2|a||b| = 2|a|^2 + 2|b|^2$

Proposition 4: The three altitudes of a triangle meet at one point.



1/ We have PQ is perpendicular to OK,

➔ $\overrightarrow{PQ}\,.\,\overrightarrow{OK} = 0$

➔ $(x_Q - x_P)(x_K - x_O) + (y_Q - y_P)(y_K - y_O) = 0$

➔ $x_Q\, x_K - x_Q\, x_O - x_P\, x_K + x_P\, x_O + y_Q\, y_K - y_Q\, y_O - y_P\, y_K + y_P\, y_O = 0$ (1)

2/ We have OQ is perpendicular to PK,

➔ $\overrightarrow{OQ}\,.\,\overrightarrow{PK} = 0$

➔ $(x_Q - x_O)(x_K - x_P) + (y_Q - y_O)(y_K - y_P) = 0$

➔ $x_Q x_K - x_Q x_P - x_O x_K + x_P x_O + y_Q y_K - y_Q y_P - y_O y_K + y_Q y_P = 0$ (2)

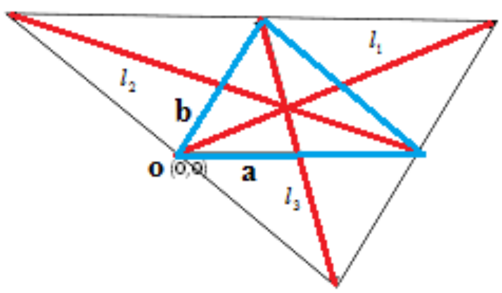
3/ Take (1) – (2), we have:

$x_Q x_P - x_Q x_O + x_O x_K - x_P x_K + y_Q y_P - y_Q y_O + y_O y_K - y_P y_K = 0$

➔ $(x_Q - x_K)(x_P - x_O) + (y_Q - y_K)(y_P - y_O) = 0$

➔ $\overrightarrow{KQ} . \overrightarrow{OP} = 0$

Proposition 5: The three medians of a triangle meet at one point.



The equations of lines: $\begin{cases} l_1 : r(t) = (a+b)t \\ l_2 : r(u) = \frac{1}{2}b + (a - \frac{1}{2}b)u \\ l_3 : r(v) = \frac{1}{2}a + (b - \frac{1}{2}a)v \end{cases}$ Solve the system equations

$t = u = v = \frac{1}{3}$ Three medians of a triangle meet at one point.

Proposition 6: The law of sines $\frac{\sin A}{|a|} = \frac{\sin B}{|b|} = \frac{sinC}{|c|}$

We have: $a + b + c = 0$ then $a \times (a + b + c) = a \times 0 = 0$ $\quad a \times a + a \times b + a \times c = 0$ since $a \times a = 0$

$a \times b - c \times a = 0$ $\quad |a||b| \sin C = |c||a| \sin B$ $\quad \frac{\sin C}{|c|} = \frac{\sin B}{|b|}$ **(1)** similarly for $a \times (a + b + c) = 0$

$b \times a - c \times b = 0$ $\quad |b||a| \sin C = |c||b| \sin A$ $\quad \frac{\sin C}{|c|} = \frac{\sin A}{|a|}$ **(2)**

From **(1)** and **(2)**, we have $\frac{\sin C}{|c|} = \frac{\sin A}{|a|} = \frac{sinB}{|b|}$

Proposition 7: If two distinct planes have a point in common, they have a line in common.

Plane (S1) has the normal vector **n** and plane (S2) has the normal vector **N**.

Let R and R_o be two random points on plane S1.

Let r and r_o be two random points on plane S2.

Let A be the point in common between two planes S1 and S2.

We have: **n** . $(R - R_o) = 0$ and **N** . $(r - r_o) = 0$

The equation of line in common: r(t) = (**n x N**) t + A

If there exists a line in common that goes through a point in common between two planes, all points on that line are also on the two planes.

We plug the equation of line in common in **N** . $(r - r_o) = 0$ (1) and **n** . $(R - R_o) = 0$ (2)

N . $(r - r_o)$ = **N** . [(**n x N**) t + A - r_o] = **N**. [(**n x N**) t] + **N**. (A - r_o) = 0 (3)

Similarly, **n** . $(R - R_o)$ = **n** . [(**n x N**) t + A - R_o] = **n**. [(**n x N**) t] + **N**. (A - R_o) = 0 (4)

0 0

From (1) & (3), and (2) & (4), we have 0 = 0 (true); therefore all points on the line in common are also on the two planes.

Conclusion: If two distinct planes have a point in common, they have a line in common.

Ori (Fold) gami (Paper) and Geometry

Axiom 1: For two points p_1 and p_2, there is a unique fold that passes through both of them.

This is equivalent to using a straight edge and compass to connect two points. (Solving a first degree equation)

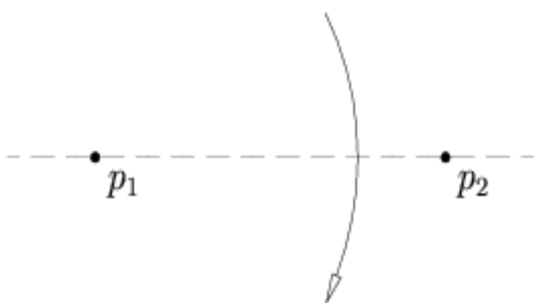


Figure 1 Axiom 1

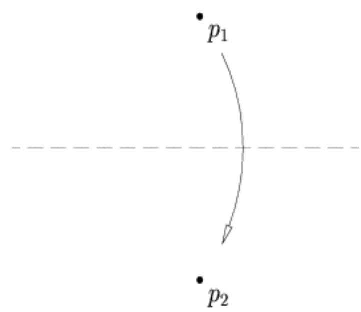


Figure 2 Axiom 2

Axiom 2: For two points p_1 and p_2, there is a unique fold that place p_1 onto p_2.

This creates the perpendicular bisector to a line that contains the two points

Axiom 3: For two lines L_1 and L_2, there is a fold that place L_1 onto L_2.

This is equivalent to bisecting the angle between the two lines

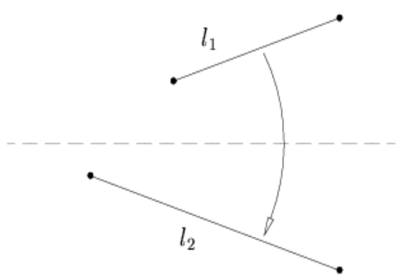


Figure 3 Axiom 3

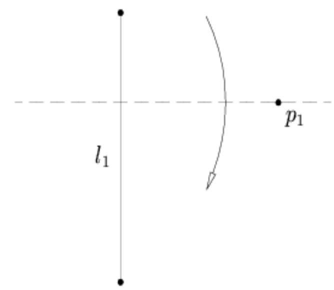


Figure 4 Axiom 4

Axiom 4: For a point p_1 and line L_1, there is a unique fold perpendicular to L_1 that passes through point p_1

This is equivalent to finding the perpendicular to a line that goes through a given point

Axiom 5: For two points p_1 and p_2 and a line L_1, there is a fold that places p_1 onto L_1 and passes through p_2.

This is equivalent to solving a second degree equation.

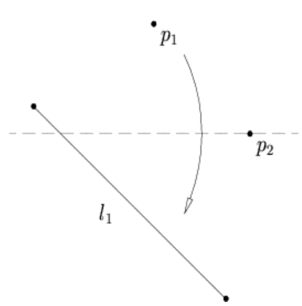


Figure 5 Axiom 5

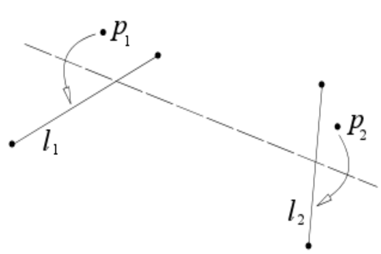
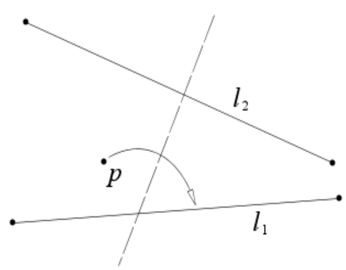


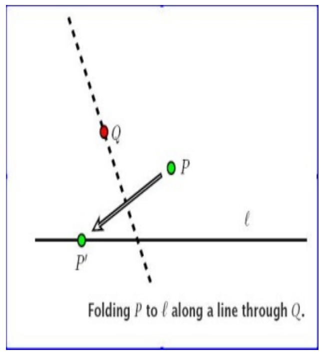
Figure 6 Axiom 6

Axiom 6: For two points p_1 and p_2 and two lines L_1 and L_2, there is a fold that places p_1 onto L_1 and p_2 onto L_2.

This is equivalent to solving a third degree equation

Axiom 7: For one point p and two line L_1 and L_2, there is a fold that places p onto L_1 and is perpendicular to L_2. (Solving a first degree equation)





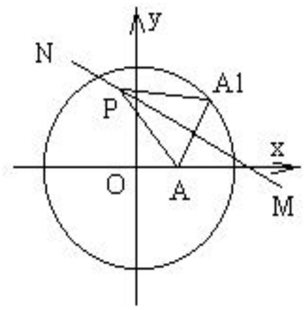


Figure 7 Axiom 7 Figure 8 Parabola Figure 9 Ellipse

Parabola: (Axiom 5 above). For two points P and Q and line L, one folds along a line through Q, taking P to a point p' on L. So if the point P is the focus and L is the directrix, then the folded line is the tangent line of the parabola.

Ellipse: A circle with radius R, center at origin with a point A on x axes. Fold the paper such that every point A_1 on the circle is the reflection point of A. Repeat the same for different point A_1 and find the collection of the points on these lines.

Kepler's Law of planetary motion

1- Every planet travels around the sun in an elliptical orbit and sun at on focus. It was discovered at 1605 and published at 1609.

2- The velocity of a planet varies in such a way that the line joining the planet to the sun sweeps out equal areas in equal time. It was discovered at 1602 and published at 1609.

3- The square of the time required by a planet for one revolution around the sun is proportional to the cube of its mean distance from the sun. It was discovered at 1618 and published at 1619.

Galileo (Italian) in 1605 was not convinced with Kepler's laws. Bullialdus (French) in 1645 asked "What cause the planets to move?"

Newton (British) in 1680 claimed that the orbits are in the form of Ellipses. Newton in 1684 with the request of astronomer Halley (British) gave a correct proof of the claim.

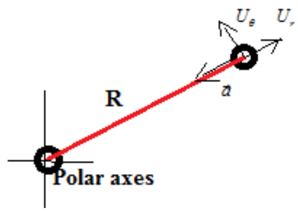
It took 2 years for newton to write Principia in 1687

Newton's law of motion

1- Every object continues in its state of rest, or of uniform motion in a straight line, unless it is compelled to change this state by force impressed upon it.

2- The Change of momentum is proportional to the motive force impressed, and is made in the direction of the line in which that force is impressed. $\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}(m\vec{V})$

$\vec{F} = \dot{m}\vec{V} + m\vec{a}$ If the mass of the object stays constant for duration of measurement then the equation reduces to $\vec{F} = m\vec{a}$ *Newton deduced his universal Laws of Gravity from Kepler's Law*

Newton's Law of gravity 

$\vec{a} \propto \frac{1}{r^2}$ and $U_r = \frac{\vec{R}}{|\vec{R}|}$ n

$\vec{a} = a_\theta U_\theta + a_r U_r$ Since the Gravitational force is along the line joining the masses then

$a_\theta = 0$ and $a_r = \frac{-K}{r^2}$

$\begin{cases} U_r =< Cos\theta, Sin\theta > \\ U_\theta =< -Sin\theta, Cos\theta > \end{cases}$ Since $U_r \bullet U_\theta = 0 \begin{cases} \frac{d}{d\theta} U_r =< -Sin\theta, Cos\theta >= U_\theta \\ \frac{d}{d\theta} U_\theta =< -Cos\theta, -Sin\theta >= -U_r \end{cases}$

$\begin{cases} \frac{d}{dt} U_r = \frac{d}{d\theta} U_r \frac{d\theta}{dt} = U_\theta \dot{\theta} \\ \frac{d}{dt} U_\theta = \frac{d}{d\theta} U_\theta \frac{d\theta}{dt} = -U_r \dot{\theta} \end{cases}$ Since $\begin{cases} \vec{R} = rU_r \\ \vec{V} = r\frac{d}{dt} U_r + \dot{r} U_r = r\dot{\theta} U_\theta + \dot{r} U_r \\ \vec{a} = (\ddot{r} - r\dot{\theta}^2) U_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta}) U_\theta \end{cases}$

The angular component of acceleration is $a_\theta = (r\ddot{\theta} + 2\dot{r}\dot{\theta}) = \frac{r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}}{r} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$

Since $a_\theta = 0$ then $\frac{d}{dt}(r^2\dot{\theta}) = 0$ and $(r^2\dot{\theta}) = Const$ then $\dot{\theta} = \frac{c}{r^2}$

$a_r = (\ddot{r} - r\dot{\theta}^2) = \frac{-K}{r^2}$ since $\dot{r} = \frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = \frac{dr}{d\theta}\frac{c}{r^2}$ Let $z = \frac{1}{r}$ $\frac{dz}{d\theta} = \frac{-1dr}{r^2 d\theta}$

With the above substitution we get $\dot{r} = -c\frac{dz}{d\theta}$ This sub eliminate r^2 from the equation

$\ddot{r} = -c\frac{d^2z}{d\theta^2}\frac{c}{r^2} = -\frac{c^2}{r^2}\frac{d^2z}{d\theta^2} = -c^2z^2\frac{d^2z}{d\theta^2}$ Rewrite $a_r = \ddot{r} - r\dot{\theta}^2 = -c^2z^2\frac{d^2z}{d\theta^2} - c^2z^3$

$a_r = -c^2z^2(\frac{d^2z}{d\theta^2} + z) = -Kz^2$ it simplifies to $\frac{d^2z}{d\theta^2} + z = \frac{K}{c^2}$. This differential equation can

be written as $\frac{d^2}{d\theta^2}(z - \frac{K}{c^2}) + (z - \frac{K}{c^2}) = 0$ then the solution of this second order differential

equation is linear combination of the solutions, in this case Sine and Cosine functions.

The solution is $z = \frac{K}{c^2} + HCos(\theta - \beta)$ Recall $ASin\theta + BCos\theta = HCos(\theta - \beta) \quad H = \sqrt{A^2 + B^2}$

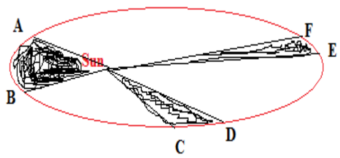
and convert it back to in terms of r, $\frac{1}{r} = \frac{K}{c^2} + HCos(\theta - \beta)$

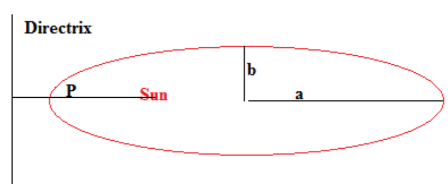
Solve for r and simplify $r = \frac{c^2}{K + Hc^2Cos(\theta - \beta)} = \frac{c^2 / K}{1 + (Hc^2 / K)Cos(\theta - \beta)}$

Recall $A = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 d\theta$ so the rate of change of area is $\frac{1}{2}r^2\dot{\theta}$

Since $r^2\dot{\theta} = Const$ then $\frac{1}{2}r^2\dot{\theta} = \frac{c}{2}$. This shows that the rate at which area is swept out by

radius to the planet.





T (Time to orbit the sun) $\frac{c}{2}$ (Rate at which area swept out) $= \pi ab$ (Area of an ellipse)

Therefor $c = \frac{2\pi ab}{T}$ and we have from the ellipse $b^2 = a^2(1-e^2)$ and $Pe = a(1-e^2)$

Then $\frac{c^2}{Pe} = (\frac{2\pi ab}{T})^2 \frac{1}{Pe} = \frac{4\pi^2}{T^2}\frac{a^2b^2}{Pe} = \frac{4\pi^2}{T^2}\frac{a^2a^2(1-e^2)}{a(1-e^2)} = \frac{4\pi^2}{T^2}a^3 = 4\pi^2\frac{a^3}{T^2}$

$\frac{T^2}{a^3} = \frac{4\pi^2}{\frac{c^2}{Pe}} = \frac{4\pi^2 Pe}{c^2} = Const$ This is the third law of planetary motion

Recall In a rotational motion $F = mR\omega^2$ for R =a $F = ma\omega^2$ which is balance by $F = \frac{GMm}{a^2}$

$F = ma\omega^2 = \frac{GMm}{a^2}$, $\omega = \frac{2\pi}{T}$ we get $(\frac{2\pi}{T})^2 = \frac{GM}{a^3}$ simplify to $\frac{T^2}{a^3} = \frac{4\pi^2}{GM}$ which is constant

Kepler knew that $\frac{T^2}{a^3}$ is constant but he did not know its value.