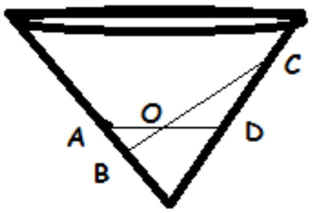
Chapter 10 Solution

1- When a plane crosses a double cones connected at their vertex, it traces the graph of conics (Ellipse, Parabola, and Hyperbola). Find a relation between the eccentricity and the angle of the plane with horizon.



Think of a cone (45degrees) is cut with a horizontal plane in a Standard x,y, and z coordinate plane. The trace of the plane on a hollow cone is a circle with radius $OD = OA = r$.

If the plane rotates with angle of θ away from xy plane, then the trace will be an ellipse with longer side $OC = L$ and shorter side $OB = S$. The sum of these two is major axis where minor axis does not change.

Look at it in 2-D: there are two triangles $\Delta OCD,\ and\ \Delta OAB$. We know the following:

$\angle DOC = \angle BOA = \theta$ and $\angle ODC = 135^\circ$, $\angle BA0 = 45^\circ$ and $\angle OCD = 45^\circ - \theta,\ \angle OBA = 135^\circ - \theta$

Use law of Sine: In ΔOCD we have $\frac{Sin(135)}{L} = \frac{Sin(45-\theta)}{r}$ and in ΔOAB $\frac{Sin(45)}{S} = \frac{Sin(135-\theta)}{r}$ Expand the sine function and simplify we have two equations

$$\begin{cases} \frac{1}{L} = \frac{Cos\theta - Sin\theta}{r} \\ \frac{1}{S} = \frac{Cos\theta + Sin\theta}{r} \end{cases} \text{ or } \begin{cases} \frac{r}{L} = Cos\theta - Sin\theta \\ \frac{r}{S} = Cos\theta + Sin\theta \end{cases}$$

Solve for Cosine and Sine in terms of L, S, and r.

$Cos\theta = \frac{r(L+S)}{2LS}\ \ and\ \ Sin\theta = \frac{r(L-S)}{2LS}$ Solve for tangent of the angle which is the slope of the line in 2D. $Tan\theta = \frac{(L-S)}{(L+S)}$

The longer side is $L = r + f$ and the shorter side is $S = r - f$ where f is the deviation of new center from old center (Focal distance).

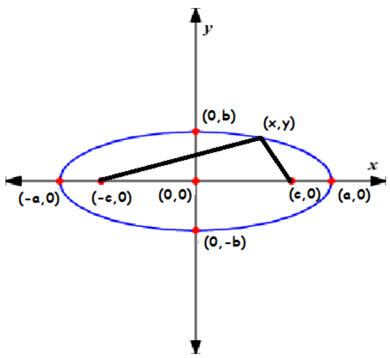
$Tan\theta = \frac{(r+f)-(r-f)}{(r+f)+(r-f)} = \frac{2f}{2r} = \frac{f}{r}$ The letter e represent this ratio or $e = \frac{f}{r} = Tan\theta$

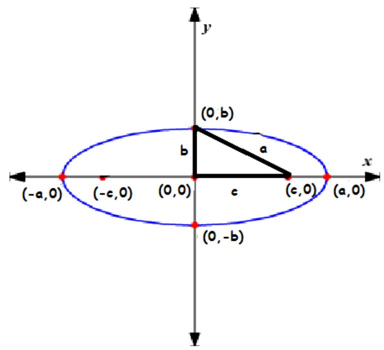
Now if $0 \le \theta < 45, or 0 \le e < 1 Ellipse \quad \theta = 45, or\ e = 1\ \ Parabola \quad \theta > 45, or\ e > 1\ \ Hyperbola$

2-Given the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ Show the following equations:

- The equation of an ellipse $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$

$$a = \frac{1}{2A}\sqrt{\frac{D^2C + E^2A - 4ACF}{C}} \quad b = \frac{1}{2C}\sqrt{\frac{D^2C + E^2A - 4ACF}{A}}$$





We begin with the drawing of the ellipse. In an ellipse, the lines going to the point (x,y) form a triangle from the focal points of the ellipse. The sum of the distances of these two lines is a constant, so we begin with the equation:

$$\sqrt{(x-(-c))^2 + (y-0)^2} + \sqrt{(x-c)^2 + (y-0)^2} = cons\tan t$$

We know that the sum of the distances must be 2a, because the lines at the x axis are the equivalent of 2a. so we get the equation:

$$\sqrt{(x-(-c))^2 + (y-0)^2} + \sqrt{(x-c)^2 + (y-0)^2} = 2a$$

Next we use algebra to derive the equation of the ellipse

$$\sqrt{(x+c)^2 + (y)^2} + \sqrt{(x-c)^2 + (y)^2} = 2a$$

$$\sqrt{(x-c)^2 + (y)^2} = 2a - \sqrt{(x+c)^2 + (y)^2}$$

$$(x+c)^2 + (y)^2 = (2a - \sqrt{(x+c)^2 + (y)^2})^2$$

$$x^2+2cx+c^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+(x-c)^2+y^2$$

$$x^2+2cx+c^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+x^2-2cx+c^2+y^2$$

$$2cx=4a^2-4a\sqrt{(x-c)^2+y^2}-2cx \qquad 4cx-4a^2=4a\sqrt{(x-c)^2+y^2}+cx-a^2=a\sqrt{(x-c)^2+y^2}$$

$$x^2+2cx+c^2+y^2=4a^2-4a\sqrt{(x-c)^2+y^2}+x^2-2cx+c^2+y^2$$

$$(cx-a^2)^2=a^2((x-c)^2+y^2)$$

$$c^2x^2+2a^2c^2x^2+a^4=a^2x^2-2a^2c^2x^2+a^2c^2+a^2y^2 \qquad c^2x^2+a^4=a^2x^2+a^2c^2+a^2y^2$$

$$a^4-a^2c^2=a^2x^2+c^2c^2+a^2y^2 \qquad a^2(a^2-c^2)=x^2(a^2-c^2)+a^2y^2$$

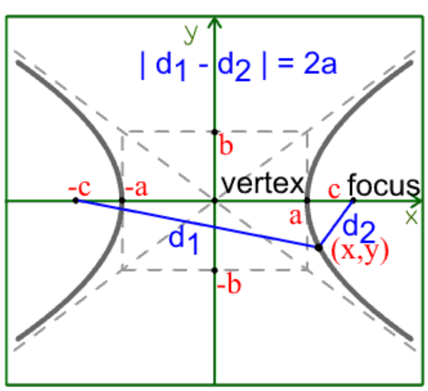
We know that $a^2=b^2+c^2$ so $a^2-c^2=b^2$ $\quad a^2b^2+b^2x^2+a^2y^2$

$$\frac{a^2b^2}{a^2b^2}=\frac{b^2x^2}{a^2b^2}+\frac{a^2y^2}{a^2b^2} \qquad 1=\frac{x^2}{a^2}+\frac{y^2}{b^2}$$

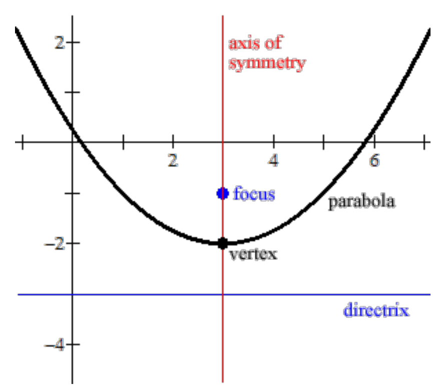
- The equation of hyperbola $\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1$

$$a=\frac{1}{2A}\sqrt{\frac{D^2C-E^2A-4ACF}{C}} \quad b=\frac{1}{2C}\sqrt{\frac{-D^2C+E^2A+4ACF}{A}}$$

Hyperbola



Parabola



The difference from (x,y) to the foci is

$$\sqrt{x-(-c)^2+(y-0)^2}+\sqrt{(x-c)^2+(y-0)^2}=2a$$

Algebraic derivation: $\sqrt{(x+c)^2+y^2}+\sqrt{(x-c)^2+y^2}=2a$

$$\sqrt{(x+c)^2+y^2}=2a-\sqrt{(x-c)^2+y^2}$$
$$(x+c)^2+y^2=\left[2a-\sqrt{(x-c)^2+y^2}\right]$$
$$x^2+2cx+c^2+y^2=4a^2-4a\sqrt{(x+c)^2+y^2}+(x+c)^2+y^2$$
$$x^2+2cx+c^2+y^2=4a^2-4a\sqrt{(x+c)^2+y^2}+x^2+2cx+c^2+y^2$$
$$2cx=4a^2-4a\sqrt{(x+c)^2+y^2}-2cx \qquad 4cx-4a^2=4a\sqrt{(x+c)^2+y^2}$$
$$cx-a^2=a\sqrt{(x+c)^2+y^2} \qquad \left(cx-a^2\right)^2=a^2\left[(x+c)^2+y^2\right]$$
$$c^2x^2-2a^2c^2x^2+a^4=a^2x^2-2a^2c^2x^2+a^2c^2+a^2y^2$$
$$c^2x^2+a^4=a^2x^2+a^2c^2+a^2y^2 \qquad a^4-a^2c^2=a^2x^2+a^2c^2+a^2y^2$$

Now we multiply by -1 to make both sides positive $a^2(a^2-c^2)=x^2(a^2-c^2)-a^2y^2$

Then substitute $b^2=c^2-a^2$ $\quad \dfrac{a^2b^2}{a^2b^2}=\dfrac{b^2x^2}{a^2b^2}-\dfrac{a^2y^2}{a^2b^2} \quad 1=\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}$

- The equation of Parabola $(y-k)=\dfrac{1}{4p}(x-h)^2$

  Where $h=(\dfrac{-D}{2A})$, $k=(\dfrac{D^2-4AF}{4AE})$ $p=\dfrac{-E}{4A}$

Parabola(The figure on top of the page)

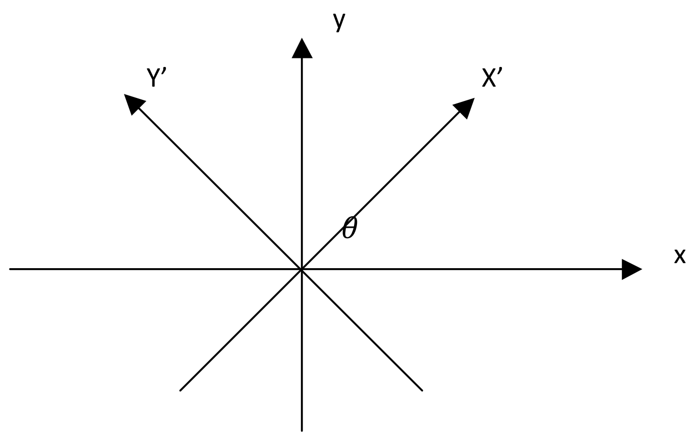
Let the vertex be at $v=(h,k)$ The directrix at $y=(k-m,m)$

The focus at $f=(h,k+m)$ And the point $p=(x,y)$

$$V\left[(x-h)^2+(y-(k+m))^2\right]=V\left[(x-x)^2+(y-(km))^2\right]$$
$$x^2-2hx+h^2+y^2-2y(k+m)+k^2+2mk+m^2=y^2-2y(k-m)+(k-m)^2$$
$$x^2-2hx+h^2+y^2-2yk-2ym+k^2+2mk+m^2=y^2-2yk-2ym+k^2-2mk+m^2$$

$$x^2-2hx+h^2-2my=2ym-2mk \qquad -4ym=-x^2+2hx-h^2-4mk$$
$$4ym=x^2-2hx+h^2+4mk \quad 4m(y-k)=(x-h)^2 \quad (y-k)=\frac{1}{4m}(x-h)^2$$

- The angle of axes rotation to eliminate the cross term which can be calculated by $Tan(2\theta) = \dfrac{B}{A-C}$



$$x = x'\cos\theta - y'\sin\theta$$
$$y = x'\sin\theta + y'\cos\theta$$

$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ (Equation of normal (x, y) plane)

$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0$ (Equation of (x, y) prime plane)

We only need to plug in x' and y' back in for 'A', 'B', and 'C' since only these terms will give us variables relating to ' x'y' '. By setting all variables relating to ' x' y ' ' we can solve for the angle of rotation.

$$A(x'\cos\theta - y'\sin\theta)^2 + B(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) + C(x'\sin\theta + y'\sin\theta)^2 = 0$$

$$A'(x'^2\cos^2\theta - 2'xy'\cos\theta\sin\theta + y^2\sin^2\theta)$$
$$+B'(x'^2\cos\theta\sin\theta + x'y'\cos^2\theta - x'y'\sin^2\theta - y'^2\cos\theta\sin\theta)$$
$$+C'(x'^2\sin^2\theta + 2x'y'\sin\theta\cos\theta + y'^2\cos^2\theta) = 0$$

We only want the xy terms so we get:

$$-A2x'y'\sin\theta\cos\theta + C2x'y'\sin\theta\cos\theta + B(x'y'\cos^2\theta - x'y'\sin^2\theta) = 0$$

Separate the variables and solve:

$$B(x'y')(\cos(2\theta)) = (A-C)(x'y')\sin(2\theta)$$

$$\tan(2\theta) = \frac{B}{A-C}$$

3-Find rectangular equation of an object launched with the following parametric equation $\begin{cases} x(t) = (vCos\theta)t + x_0 \\ y(t) = -4.9t^2 + (vSin\theta)t + y_0 \end{cases}$ if it launched from origin, ***use the rectangular equation*** to find a) Maximum Range b) Maximum height. Is it possible to find velocity of impact? Find it.

We have $x(t) = (v\cos\theta)t + x_0$ and $y(t) = -4.9t^2 + (v\sin\theta)t + y_0$

Since it is launched at the origin, equals 0. Then solve for t:

Plug in t into y(t):

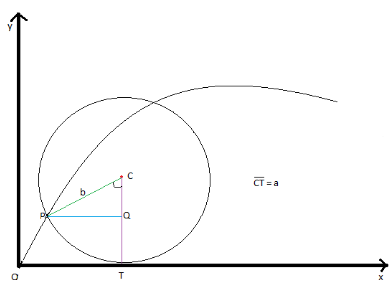
Now set y = 0 to find the maximum range (x - intercept). Solve for t

Differentiate y to find the maximum height (when y' = 0)

$$y_{\max} = \frac{v^2 \sin^2\theta}{2g}$$

4-Show that the parametric equation of all three different Cycloid can be written as $\begin{cases} x(t) = a\theta - bSin\theta \\ y(t) = a - bCos\theta \end{cases}$ Then find the slope of the curve in terms of angle θ.

be written as $\begin{cases} x(t) = a\theta - bSin\theta \\ y(t) = a - bCos\theta \end{cases}$. Then find the slope in terms of angle θ.



Find point P as reference for the parametric equation.

$\overline{OT} = \overline{PQ}$, so length of line $\overline{OT} = a\theta$

To find the x variable of point P, we will subtract line $\overline{OT}$ from line $\overline{PQ}$ which turns out to be:

$x(t) = a\theta - b\sin\theta$

To find the y variable of point P, we will find $\overline{QT}$.

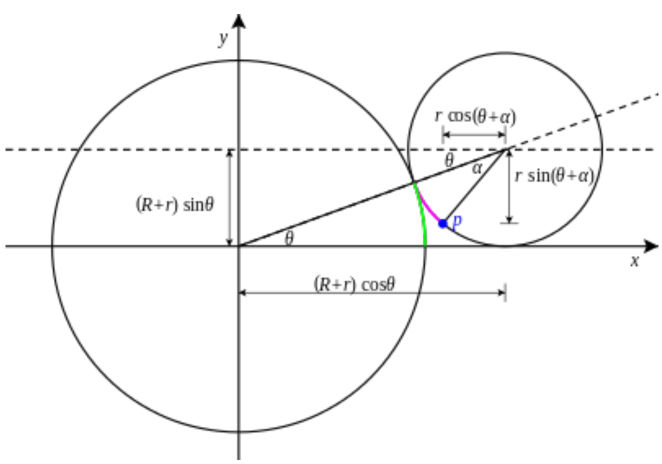
- $y(t) = a - b\cos\theta$
- When $\frac{b}{a} > 1$, this equation will yield an epicycloid.
- When $\frac{b}{a} = 1$, this equation will yield a standard cycloid.
- When $\frac{b}{a} < 1$, this equation will yield a trochoid.

Slope at any point on this graph will be denoted by:

$$M = \frac{dy}{d\theta} \div \frac{dx}{d\theta} = \frac{b\sin\theta}{a - b\cos\theta}$$

5-Show that the parametric equation of all epicycle can be written as

$$\begin{cases} x(t) = (a+b)Cos\theta - bCos(\frac{a+b}{b}\theta) \\ y(t) = (a+b)Sin\theta - bSin(\frac{a+b}{b}\theta) \end{cases}$$



We assume that the position of p is what we want to solve, α is the radian from the tangential point to the moving point p, and θ is the radian from the starting point to the tangential point.

Since there is no sliding between the two cycles, then we have that

$$L_R = L_r$$

By the definition of radian (which is the rate arc over radius), then we have that

$L_R = \theta R \;\; ; \;\; L_r = \alpha r$

From these two conditions, we get the identity

$\theta R = \alpha r$

By calculating, we get the relation between α and θ, which is

$$\alpha = \frac{R}{r}\theta$$

From the figure, we see the position of the point P clearly.

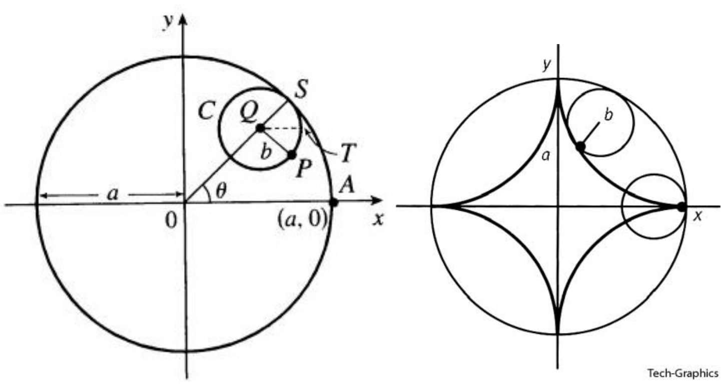
$$x = (R+r)\cos\theta - r\cos(\theta+\alpha) = (R+r)\cos\theta - r\cos\left(\frac{R+r}{r}\theta\right)$$

$$x = (R+r)\sin\theta - r\sin(\theta+\alpha) = (R+r)\sin\theta - r\sin\left(\frac{R+r}{r}\theta\right)$$

6- Show that the parametric equation of all hypocycloid can be written as

$$\begin{cases} x(t) = (a-b)Cos\theta + bCos(\frac{a-b}{b}\theta) \\ y(t) = (a-b)Sin\theta - bSin(\frac{a-b}{b}\theta) \end{cases}$$

A **hypocycloid** is a curve traced out by a fixed point *P* on a circle *C* of radius *b* as *C* rolls on the inside of a circle with center *O* and radius *a*.



If the point, P, starts at (a,0) then Arc PS = Arc As. Thus, ∠ PQS = (a/b) θ and ∠ PQT = (a/b) θ - θ. The center of the smaller circle, Q, has the coordinates

((a-b)Cosθ, (a-b)Sinθ) So, P has the coordinates

X = (a-b)cosθ + bcos(∠ PQT) = (a-b)cosθ + bcos((a/b) θ - θ)

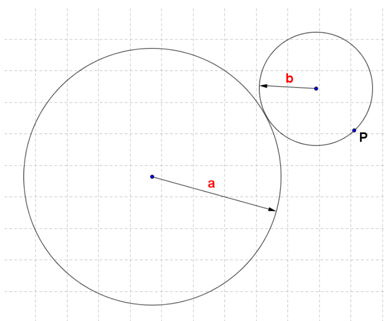
Y = (a-b)sinθ + bsin(∠ PQT) = (a-b)sinθ + bsin((a/b) θ - θ)

Which simplifies to

$$\begin{cases} x(t) = (a-b)Cos\theta + bCos(\frac{a-b}{b}\theta) \\ y(t) = (a-b)Sin\theta - bSin(\frac{a-b}{b}\theta) \end{cases}$$

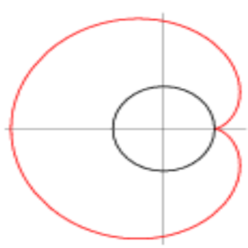
7-Show that polar graph of Rose and cardioid can be trace by epicycles

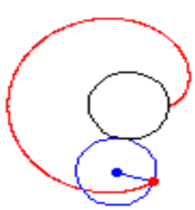
Epicycle is about one center circle and the other circle moving around it.



The path traced out by a point P on the edge of a circle of radius a = b.

The radiuses of both circles are the same so it creates a cardioid shape.

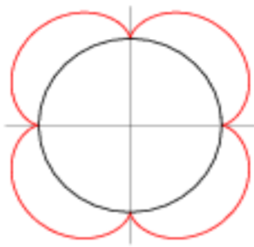


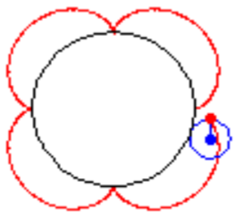


The path traced out by a point P on the edge of a circle of radius a = 4b

The radius of the center circle greater than the outside 4 times.

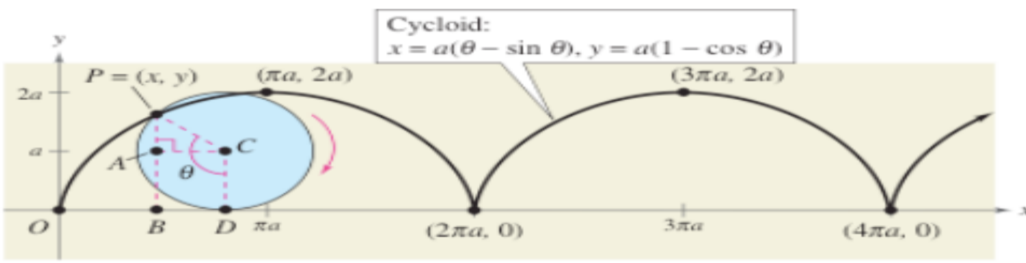
It is called a rose traced by epicycle.





Problem 8,9,10,and 11 are about Cycloids

8-Graph a cycloid generated by a disc with radius one, then find the tangent line to the curve at $\theta = \pi/2$.



$$\sin\theta = \sin(180-\theta) = \sin(\angle APC) = \frac{AC}{a} = \frac{BD}{a} \qquad \frac{dx}{d\theta} = 1-\cos\theta$$

$$\cos\theta = -\cos(180-\theta) = -\cos(\angle APC) = -\frac{AP}{a} \qquad \frac{dy}{d\theta} = \sin\theta$$

$$x = OD - BD = a\theta - a\sin\theta \qquad y = BA + AP = a - a\cos\theta \qquad \frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta}$$

$$\boxed{\begin{aligned} x &= a(\theta - \sin\theta) \\ y &= a(1-\cos\theta) \end{aligned}}$$

9- Find arc length and the area under the curve of a cycloid after one rotation of generating disc.

Solutions: Parametric equations of cycloid are $x = r\theta - r\sin\theta$, $y = r - r\cos\theta$

$$\frac{dx}{d\theta} = r - r\cos\theta = r(1-\cos\theta),\ \ \frac{dy}{d\theta} = r\sin\theta$$

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{r^2(1-\cos\theta)^2 + r^2\sin^2\theta}\, d\theta$$

$$r = \int_0^{2\pi} \sqrt{1-2\cos\theta) + \left(\cos^2\theta + \sin^2\theta\right)}\, d\theta = r\int_0^{2\pi} \sqrt{(2-2\cos\theta)}\, d\theta = r\int_0^{2\pi} \sqrt{2(1-\cos\theta)}\, d\theta$$

Since $\sin^2\theta = \frac{1-\cos 2\theta}{2}$, $\sqrt{2(1-\cos\theta)} = \sqrt{4\sin^2\left(\frac{\theta}{2}\right)} = 2\sin\left(\frac{\theta}{2}\right)$

So $L = 2r\int_0^{2\pi} \sin\left(\frac{\theta}{2}\right) d\theta = 2r\left[-2\cos\left(\frac{\theta}{2}\right)\right]_0^{2\pi} = 8r$

$$A = \int_0^{2\pi} y\,dx = \int_0^{2\pi} r(1-\cos\theta)(r - r\cos\theta)\,d\theta = r^2\int_0^{2\pi}(\cos^2\theta - 2\cos\theta + 1)\,d\theta$$

$$= r^2\int_0^{2\pi}\left(\frac{1+\cos 2\theta}{2} - 2\cos\theta + 1\right)d\theta$$

$$= r^2\int_0^{2\pi}\left(\frac{\cos 2\theta}{2} - 2\cos\theta + \frac{3}{2}\right)d\theta = r^2\left[\frac{\sin 2\theta}{4} - 2\sin\theta + \frac{3}{2}\theta\right]_0^{2\pi} = r^2\left(2\pi\cdot\frac{3}{2}\right) = 3\pi r^2$$

10- Find Surface area of rotating a cycloid after one rotation of generating disc around x axis.

Solution- General cycloid equation is

$$\begin{cases} x(t) = a(\theta - Sin\theta) \\ y(t) = a(1 - Cos\theta) \end{cases}$$

(a is the radius of the circle)

First, think about Δs, which is the surface of ring with radius = y, width

$= \sqrt{(\Delta x)^2 + (\Delta y)^2}$ $\quad \Delta S \approx 2\pi y\sqrt{(\Delta x)^2 + (\Delta y)^2}$ *let domain of definition*

Be $0 \le \theta < 2\pi$ then surface area of rotating a cycloid is going to be

$$S \approx \int_0^{2\pi} 2\pi y\sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} 2\pi\{a(1-\cos\theta)\}\sqrt{\{a(1-\cos\theta)\}^2 + \{a(\sin\theta)\}^2}\, d\theta$$

$$= 2\pi a^2 \int_0^{2\pi} (1-\cos\theta)\sqrt{(1-\cos\theta)^2 + (\sin\theta)^2}\, d\theta = 2\pi a^2 \int_0^{2\pi} (1-\cos\theta)\sqrt{2(1-\cos\theta)}\, d\theta$$

$$= 2\pi a^2 \int_0^{2\pi} 2\left(\sin\left(\frac{\theta}{2}\right)\right)^2 \sqrt{4\left\{\sin\left(\frac{\theta}{2}\right)\right\}^2}\, d\theta = 8\pi a^2 \int_0^{2\pi} \left(\sin\left(\frac{\theta}{2}\right)\right)^2 \left|\sin\left(\frac{\theta}{2}\right)\right| d\theta$$

$$= 2\pi a^2 \int_0^{2\pi} 2(\sin\theta)\sqrt{(1-\cos\theta)^2 + (\sin\theta)^2}\, d\theta = 2\pi a^2 \int_0^{2\pi} (1-\cos\theta)\sqrt{2(1-\cos\theta)}\, d\theta$$

$Let\ \ t = \frac{\theta}{2}, then\ \ d\theta = 2dt, and\ \ \theta:\ 0\ \ to\ \ 2\pi, t = 0\ \ to\ \ \pi.\ \ S = 8\pi a^2 \int_0^{\pi} \{\sin(t)\}^2 (\sin(t))(2dt)$

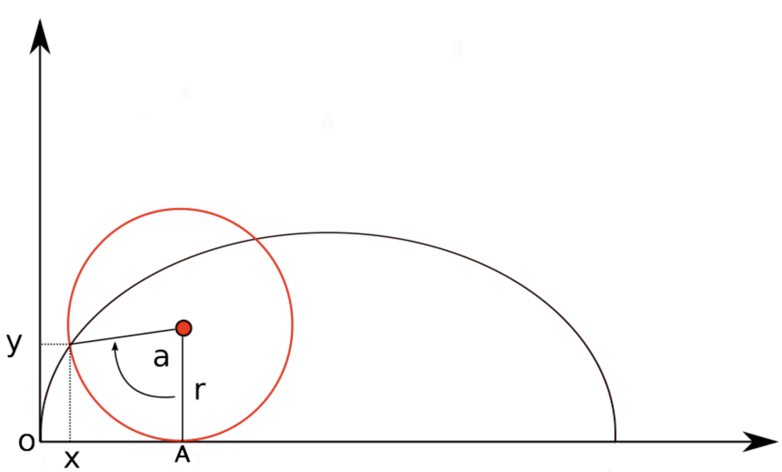
$$= 16\pi a^2 \int_0^{\pi} \{\sin(t)\}^3 (dt) = 32\pi a^2 \int_0^{\frac{\theta}{2}} \{\sin(t)\}^2 (\sin(t))dt$$

$$= 32\pi a^2 \int_0^{\frac{\theta}{2}} (1-\cos^2(t))^2 (\sin(t))dt$$

By substation rule, cos(t)=u, then du= -sin(t)dt. And t: 0 to $\frac{\theta}{2}$, u: 1 to 0.

$$= 32\pi a^2 \int_1^0 (\sin(t))(1-u^2)\left(-\frac{du}{\sin(t)}\right) = -32\pi a^2 \int_0^1 (u^2-1)du = -32\pi a^2 \left[\frac{1}{3}u^3 - u\right]_0^1 = \frac{64}{3}\pi a^2$$

11- If you double the radius of generating disc, by what factors the Arc length, the Area and the Surface area changes?



$cycloid \quad with \quad normal \quad radius:$

$x = r(a - \sin a)$

$y = r(1 - \cos a)$

$$S = \int_0^{2\pi} \sqrt{r^2(1-\cos a)^2 + r^2 \sin^2 a}\,da = \int_0^{2\pi} \sqrt{2r^2(1-\cos a)}\,da = \int_0^{2\pi} \sqrt{4r^2 \sin^2\left(\frac{a}{2}\right)}\,da = 2r\int_0^{2\pi} \sin\left(\frac{a}{2}\right)da = 8r$$

$$SA = 2\pi \int_0^{2\pi} r(1-\cos a)\sqrt{r^2(1-\cos a)^2 + r^2 \sin^2 a}\,da = 2\pi \int_0^{2\pi} r(1-\cos a)2r\sin\left(\frac{a}{2}\right)da$$

$$= 2\pi \int_0^{2\pi} 2r^2 \sin^3\left(\frac{a}{2}\right)da \to 4r^2 \int_0^{2\pi} \sin^3\left(\frac{a}{2}\right)da \qquad t = \frac{a}{2}$$

$$= 4r^2\pi \int_0^{\pi} \sin^3 t\,dt = 4r^2\pi\left(\frac{\cos^3 t}{3} - \cos t\right)\Bigg|_0^{\pi} = \frac{16}{3}r^2\pi$$

$$A = \int_0^{2\pi} r(1-\cos a)r(1-\cos a)\,da = \int_0^{2\pi} r^2(1-\cos a)^2\,da = r^2\int_0^{2\pi}(1 - 2\cos a + \cos^2 a)\,da$$

$$= r^2\left(\frac{3}{2}a - 2\sin a + \frac{1}{4}\sin(2a)\right)\Bigg|_0^{2\pi} = 3r^2$$

Cycloid with doubled radius :

$x = 2r(a - \sin a)$

$y = 2r(1 - \cos a)$

$$A = \int_0^{2\pi} \sqrt{4r^2(1-\cos a)^2 + 4r^2 \sin^2 a}\,da$$

$$= 2r\int_0^{2\pi} \sqrt{2(1-\cos a)}\,da$$

$$= 4r\int_0^{2\pi} \sqrt{\sin^2(\frac{a}{2})}\,da$$

$$= 4r(-2\cos(\frac{a}{2}))\Big|_0^{2\pi}$$

$$= 16r$$

$$SA = 2\pi\int_0^{2\pi} 2r(1-\cos a)\sqrt{4r^2(1-\cos a)^2 + 4r^2\sin^2 a}\, da$$

$$= 2\pi\int_0^{2\pi} 4r^2(1-\cos a)\sqrt{2(1-\cos a)}\, da$$

$$= 2\pi\int_0^{2\pi} 4r^2(1-\cos a)\sqrt{4(\sin^2\frac{a}{2})}\, da$$

$$= 4\pi\int_0^{2\pi} 4r^2(1-\cos a)(\sin\frac{a}{2})\, da$$

$$= 4\pi\int_0^{2\pi} 8r^2(\sin^3\frac{a}{2})\, da$$

$$= 16\pi r^2\int_0^{\pi} \sin^3 u\, du$$

$$= 16\pi r^2(\frac{\cos^3 u}{3} - \cos u)$$

$$= \frac{128}{3}\pi r^2$$

$$A = \int_a^b y(t)(\frac{dx}{dt})\, dt \ = \int_0^{2\pi} (2r - 2r\cos a)(2r - 2r\cos a)\, da$$

$$= \int_0^{2\pi} (2r - 2r\cos a)(2r - 2r\cos a)\, da$$

$$= \int_0^{2\pi} (4r^2 - 8r^2\cos a + 4r^2\cos^2 a)\, da$$

$$= \int_0^{2\pi} 4r^2(1 - 2\cos a + \cos^2 a)\, da$$

$$= 4r^2\int_0^{2\pi} da - 8r^2\int_0^{2\pi}\cos a\, da + 4r^2\int_0^{2\pi}\cos^2 a\, da$$

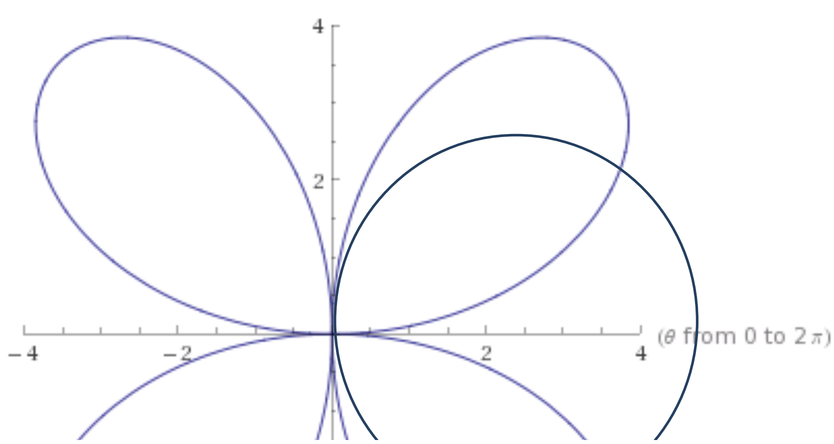
$$= 8\pi r^2 + 4\pi r^2$$

$$= 12\pi r^2$$

12- Find the area inside both functions. (*Graph them*)

$r_1 = Sin(2\theta)$

$r_2 = Cos\theta$



When graphing the equations, we obtain the graph above.

To find the area between the curves, we must first calculate the points of intersection. To do this we set the equations of the curves equal to each other and solve for :

$$\sin(2\theta) = \cos(\theta)$$
$$\rightarrow \sin(2\theta) - \cos(\theta) = 0$$
$$\rightarrow 2\sin(\theta)\cos(\theta) - \cos(\theta) = 0$$
$$\rightarrow \cos(\theta)(2\sin(\theta) - 1) = 0$$

$$\cos(\theta) = 0 \quad ; \quad 2\sin(\theta) - 1 = 0$$
$$\therefore \quad \theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{\pi}{2}, \frac{3\pi}{2}$$

Now we know the angles at which the graphs intersect. Based on these values, we obtain our limits of integration.

We know that to find the area of a polar function we must use the equation:

$$A = \frac{1}{2}\int_a^b r^2 d\theta$$

We also know that since the graphs are symmetrical across the polar axis we can take the integral from 0 radians to π radians and multiply the integral by 2 to find the total area between the two curves.

Therefore our equation would look like this:

$$A = 2\left[\frac{1}{2}\int_0^{\pi/6}[\cos(\theta) - \sin(2\theta)]^2 d\theta\right]$$

To find the area between two curves you must subtract the innermost function from the outermost function. This is why we subtract the sin(2θ) from the cos(θ).

Then we simplify:

$$A = 2\left[\frac{1}{2}\int_0^{\pi/6}[\cos(\theta) - \sin(2\theta)]^2\, d\theta\right]$$

$$= 2\left[\frac{1}{2}\int_0^{\pi/6}\left[\cos^2(\theta) - 2\sin(2\theta)\cos(\theta) + \sin^2(2\theta)\right]d\theta\right]$$

*This integral is unbelievably long and would take an hour for me just to type it all up so I will leave that you your mathematical imagination! ☺

But basically, the area between the two curves would be represented by the integral above

13- Find the tangent of the angle between the tangent lines of both graphs at the point of intersection.

$$r_1 = \sin(2\theta) \to r_1' = 2\cos(2\theta)$$

$$\frac{dy}{dx} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \frac{2\cos(2\theta)\sin\theta + \sin(2\theta)\cos\theta}{2\cos(2\theta)\cos\theta - \sin(2\theta)\sin\theta}$$

$$at \quad \frac{\pi}{6} \to \frac{2\cos\left(2\frac{\pi}{6}\right)\sin\left(\frac{\pi}{6}\right) + \sin\left(2\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}\right)}{2\cos\left(2\frac{\pi}{6}\right)\cos\left(\frac{\pi}{6}\right) - \sin\left(2\frac{\pi}{6}\right)\sin\left(\frac{\pi}{6}\right)}$$

$$= \frac{\frac{1}{2} + \frac{3}{2}}{\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{4}} = \frac{\frac{4}{2}}{\frac{\sqrt{3}}{4}} = \frac{8}{\sqrt{3}}$$

$$r_2 = \cos\theta \to r_2' = -\sin\theta$$

$$\frac{dy}{dx} = \frac{r'\sin\theta + r\cos\theta}{r'\cos\theta - r\sin\theta} = \frac{-\sin\theta\sin\theta + \cos\theta\cos\theta}{-\sin\theta\cos\theta - \cos\theta\sin\theta}$$

$$=\frac{\left(\frac{8}{\sqrt{3}}-\frac{1}{\sqrt{3}}\right)}{1-\left(\frac{8}{\sqrt{3}}\right)\left(\frac{-1}{\sqrt{3}}\right)}=\frac{\frac{7}{\sqrt{3}}}{1+\frac{8}{3}}=\frac{\frac{7}{\sqrt{3}}}{\frac{11}{3}}$$

$$\rightarrow 7\sqrt{3}\times\frac{3}{11}=\frac{21\sqrt{3}}{11}$$

$$\tan(\beta)=\frac{21\sqrt{3}}{11}$$

$$\beta=\tan^{-1}\left(\frac{21\sqrt{3}}{11}\right)\approx 73.173 \quad or \quad 73°10'24.78''$$

14- Find the arc length for the parametric equation

$$\begin{pmatrix} x(t)=\sin t \\ y(t)=1-\cos^2 t \end{pmatrix} \quad 0\le t\le\frac{\pi}{2}$$

$$\int_0^{\frac{\pi}{2}}\cos^2 t+(\sin 2t)^2\,dt \rightarrow \int_0^{\frac{\pi}{2}}\cos^2 t\,dt \rightarrow \int_0^{\frac{\pi}{2}}\frac{1+\cos 2t}{2}\,dt \rightarrow \left[\frac{1}{2}t+\frac{1}{4}\sin 2t\right]_0^{\frac{\pi}{2}}=\frac{\pi}{2}$$

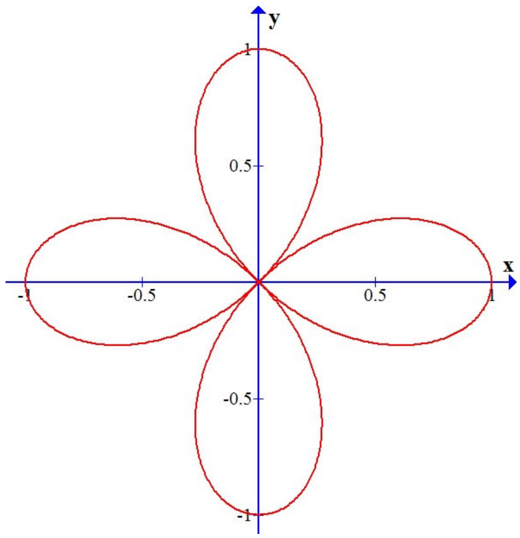
$$\int_0^{\frac{\pi}{2}}(\sin 2t)^2\,dt \rightarrow \frac{1}{2}\int_0^{\frac{\pi}{2}}(\sin u)^2\,du \rightarrow \frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{1-\cos 2u}{2}\,du \rightarrow \left[\frac{1}{2}t+\frac{1}{8}\sin 4t\right]_0^{\frac{\pi}{2}}=\frac{\pi}{2}$$

$$\rightarrow \int_0^{\frac{\pi}{2}}\cos^2 t+(\sin 2t)^2\,dt=\pi$$

15-Given the polar function $r=Sin(t)$ and line $y=y_0 \ where \ y_0<0$

What must be y_0 such that the surface area of the function rotated around $y=y_0$ is three times bigger than the surface area rotated around y = 0.

16- Graph the function $r(\theta)=Cos(2\theta)$. Show that no part of the curve overlaps with itself.



If the graph never overlaps at any point, that would mean the points (0, π/4), and (0, 5π/4) both would have to have the same slope.

General equation to find slope of parametric curves:

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\frac{dr}{d\theta}\sin\theta + \cos\theta}{\frac{dr}{d\theta}\sin\theta + \cos\theta}$$

But since r= 0 at points (0, π/4) and (0, 5π/4), this equation simplifies to

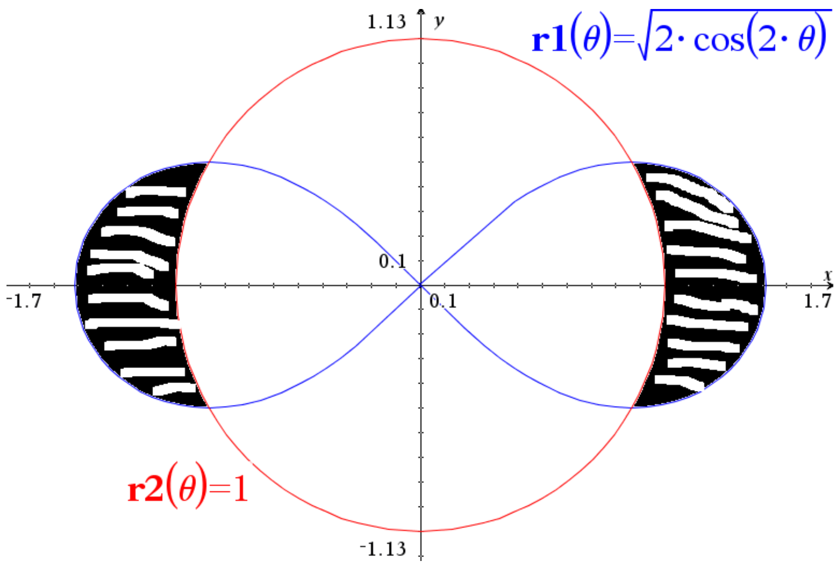
$$\frac{dy}{dx} = \tan\theta$$

Plugging in the two points we find that

$$\frac{dy}{dx} = \tan\theta = \tan\left(\frac{\pi}{4}\right) = 1$$
$$\frac{dy}{dx} = \tan\theta = \tan\left(\frac{5\pi}{4}\right) = 1$$

The slope at both points is equal to 1, thus the curve never overlaps.

17-Find the total area inside of curve $r^2 = 2Cos(2\theta)$ and outside $r = 1$



Before even thinking about how to solve for the area, we must consider the angles of intersection of the functions in order to determine an interval for which to integrate over.

This can be done by equating the magnitudes of functions and solving for the angles. $r_1 = \sqrt{2\cos(2\theta)}$
$r_2 = 1$

Equating the functions results in $\sqrt{2\cos(2\theta)} = 1$

Squaring both sides of the equation gives us $2\cos(2\theta) = 1$

Dividing both sides of the equation results in $\cos(2\theta) = \frac{1}{2}$

If $\theta = \frac{\pi}{3}$ for the $\cos(\theta) = \frac{1}{2}$, then $\theta = \frac{\pi}{6}$ for the $\cos(2\theta) = \frac{1}{2}$

Due to the periodic nature of the cosine functions, we can find all of the angles where cosine is ½ using the equations

$$\theta = \frac{\pi}{6} + n\pi$$
$$\theta = -\frac{\pi}{6} - n\pi$$

The angles of intersection are as $-\frac{\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}, and \frac{7\pi}{6}$

Now that we know where the two functions intersect each other we can calculate the area by utilizing the integral of $\int \frac{1}{2}(r)^2 d\theta$

The Area can then be written as

$$A=\frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} 2\cos(2\theta)d\theta-\frac{1}{2}\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} d\theta+\frac{1}{2}\int_{\frac{5\pi}{6}}^{\frac{7\pi}{6}} 2\cos(2\theta)d\theta-\frac{1}{2}\int_{\frac{5\pi}{6}}^{\frac{7\pi}{6}} d\theta$$

These are simple integrals whose integrations are trivial

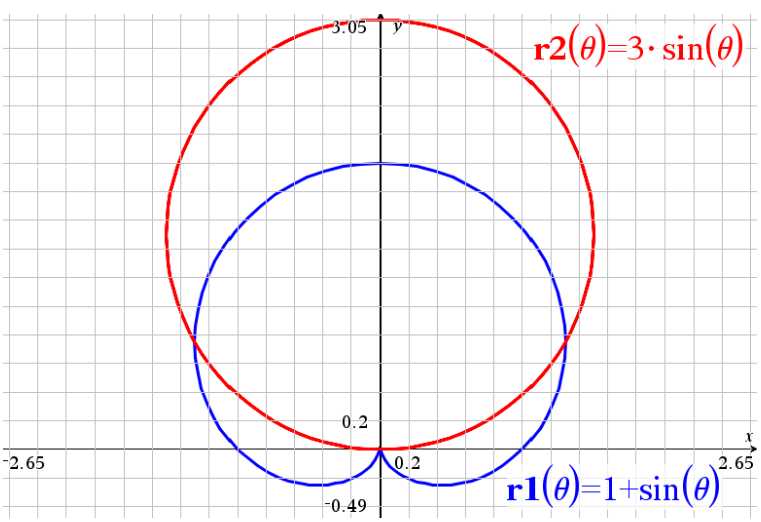
$$A=\left[\frac{1}{2}\sin 2\theta\right]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}-[\theta]_{-\frac{\pi}{6}}^{\frac{\pi}{6}}+\left[\frac{1}{2}\sin 2\theta\right]_{\frac{5\pi}{6}}^{\frac{7\pi}{6}}-[\theta]_{\frac{5\pi}{6}}^{\frac{7\pi}{6}}$$

$$A=2\left(\frac{\sqrt{3}}{2}+\frac{\pi}{6}\right)$$

Alternatively, if one is keen on the symmetry present within the functions, the Area can be expressed as

$$A=\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} 2\cos(2\theta)d\theta-\int_{-\frac{\pi}{6}}^{\frac{\pi}{6}} d\ \theta$$ Which will also yield $A=2\left(\frac{\sqrt{3}}{2}+\frac{\pi}{6}\right)$

18-Find the total area inside of curve $r=1+\sin\theta$ and outside $r=3\sin\theta$



a) First Find the points of intersection

$$1+\sin\theta = 3\sin\theta$$

$$\frac{1}{2} = \frac{2\sin\theta}{2} \therefore \sin\theta = \frac{1}{2}$$

$$\therefore \ \ the \ \ two \ \ curves \ \ intersect \ \ at \ \ \frac{\pi}{6} \ \ and \ \ at \ \ \frac{5\pi}{6}$$

b) The area of the desired region will be the (Area of the Cardioid $r = 1+\sin\theta$) – (Area of the Circle $r = 3\sin\theta$) with limits from $\frac{5\pi}{6}$ *to* $\frac{\pi}{6}$.

$$A = \int_{\frac{5\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2}(1+\sin\theta)^2\, d\theta - \int_{\frac{5\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2}(3\sin\theta)^2\, d\theta$$

(Note: because both curves are symmetric of the vertical line ($\frac{\pi}{2}$), the area curve can be written as follows:)

$$A = 2\left[\frac{1}{2}\int_{\frac{5\pi}{6}}^{\frac{3\pi}{6}} (1+\sin\theta)^2\, d\theta - \frac{1}{2}\int_{\frac{5\pi}{6}}^{\frac{3\pi}{6}} (3\sin\theta)^2\, d\theta\right]$$

$$A = \int_{\frac{5\pi}{6}}^{\frac{3\pi}{6}} \left(1+2\sin\theta+\sin^2\theta\right)d\theta - \int_{\frac{5\pi}{6}}^{\frac{3\pi}{6}} \left(9\sin^2\theta\right)d\theta$$

$$A = \int_{\frac{5\pi}{6}}^{\frac{3\pi}{6}} \left(1+2\sin\theta-8\sin^2\theta\right)d\theta \quad \left[note: \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)\right]$$

$$A = \int_{\frac{5\pi}{6}}^{\frac{3\pi}{6}} \left(4\cos 2\theta + 2\sin\theta - 3\right)d\theta \xrightarrow{YIELDS} A = \left[2\sin 2\theta - 2\cos\theta - 3\theta\right]_{\frac{5\pi}{6}}^{\frac{3\pi}{2}}$$

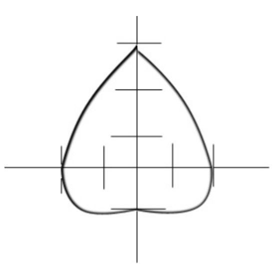
$$A = \left[0 - 0 - \frac{\pi}{2}\right] - \left[-\sqrt{3} - 2 - \frac{\pi}{2}\right]$$

$$A = 2+\sqrt{3} \xrightarrow{YIELDS} A = 3.732 \ \ (units)^2$$

$$A = \int_{\frac{5\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2}\left(1+\sin\theta\right)^2 d\theta - \int_{\frac{5\pi}{6}}^{\frac{\pi}{6}} \frac{1}{2}\left(3\sin\theta\right)^2 d\theta$$

19-Find the total area inside of the loop of $r = 2 + Sin(\theta)$

-Graph:



$$A = (1/2)\int r^2 d\theta$$

$$A = (1/2)\int_{-\pi/2}^{\pi/2} (2+\sin\theta)^2 d\theta$$

$$A = (1/2)\int_{-\pi/2}^{\pi/2} (4+\sin^2\theta + 4Sin\theta)d\theta$$

$$A = 2[(1/2)[\int_{-\pi/2}^{\pi/2} 4d\theta + \int_{-\pi/2}^{\pi/2} \frac{1-\cos 2\theta}{2} d\theta + 4\int_{-\pi/2}^{\pi/2} \sin\theta d\theta]]$$

$$A = [4\theta + ((1/2)\theta - (1/4)Sin\theta) + (-4Cos\theta) + C]_{-\pi/2}^{\pi/2}$$

$$A = 4\pi$$

20-Given the equation of curvature in polar to be $\kappa = \frac{|2r'^2 - rr'' + r^2|}{[r'^2 + r^2]^{3/2}}$

$$r = \sin\theta$$
$$r' = \cos\theta$$
$$r'' = -\sin\theta$$

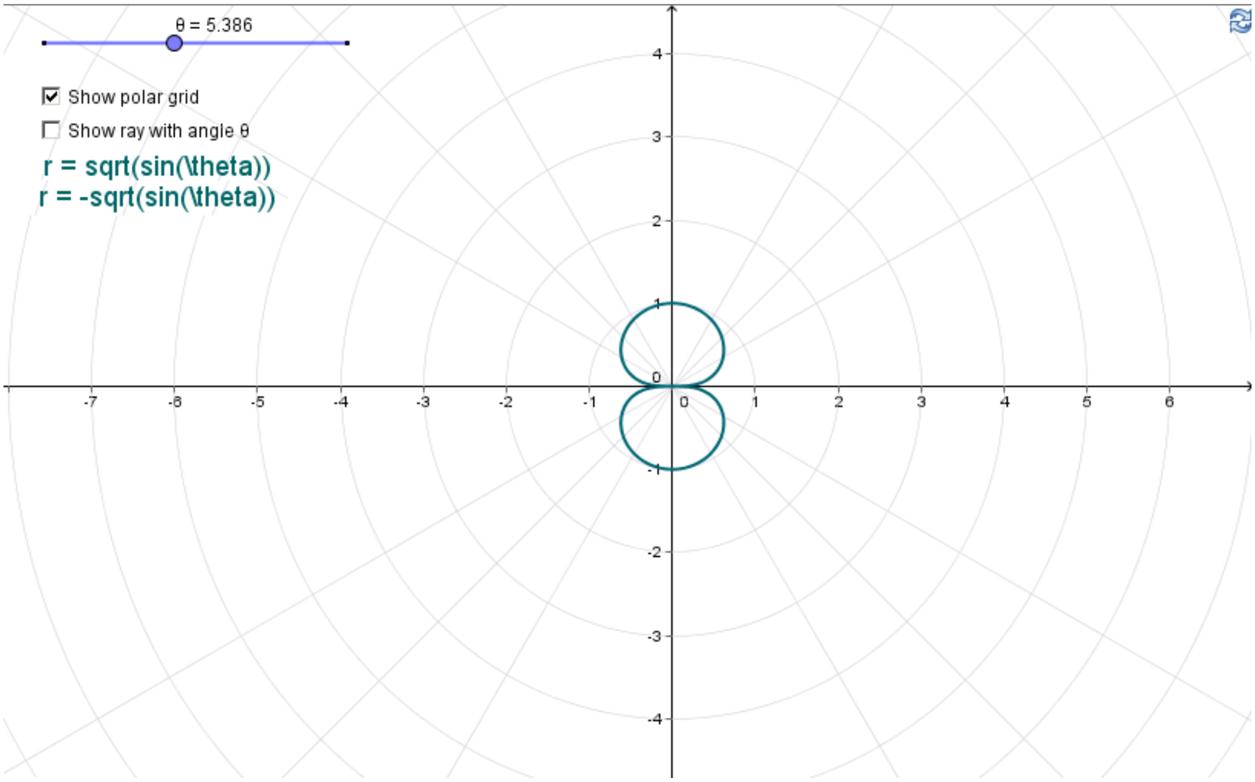
$$\kappa = \frac{2\cos^2\theta + \sin^2\theta + \sin^2\theta}{[\cos^2\theta + \sin^2\theta]^{3/2}}$$

$$\kappa = 2(\cos^2\theta + \sin^2\theta)$$

$$\kappa = \frac{2\cos^2\theta + 2\sin^2\theta}{1}$$

$$k = 2$$

21-Find all the VERTICAL and HORIZONTAL slopes to the curve $r^2 = \sin\theta$



a. Convert to rectangular

$r^2 = \sin\theta$ Given

$r^3 = r\sin\theta$ Multiply r to both sides

$r^3 = y$ Substitution

$r = \sqrt{x^2 + y^2}$ Pythagorean Theorem

$y = (x^2 + y^2)^{\frac{3}{2}}$ Substitution

$y^{\frac{2}{3}} = x^2 + y^2$ Algebra

$y^{\frac{2}{3}} - y^2 = x^2$ Algebra

$\left(\frac{2}{3}y^{-\frac{1}{3}} + 2y\right)\frac{dy}{dx} = 2x$ Derive

$$\frac{dy}{dx} = \frac{2x}{\frac{2}{3}y^{-\frac{1}{3}} + 2y}$$ Simplification

b. Find conditions for $\frac{dy}{dx} = 0$ (HORIZONTAL)

all values where x=0;

$y^{\frac{2}{3}} - y^2 = 0$ Substitution

$y^{\frac{2}{3}} = y^2$ Algebra

y=-1,y=0,y=1

c. Find conditions for $\frac{dy}{dx} = \infty$ (VERTICAL)

all values where $\frac{2}{3}y^{-\frac{1}{3}} + 2y$ =0

$\frac{2}{3}y^{-\frac{1}{3}} = -2y$ Algebra

$\frac{2}{3} = -2y(y^{\frac{1}{3}})$

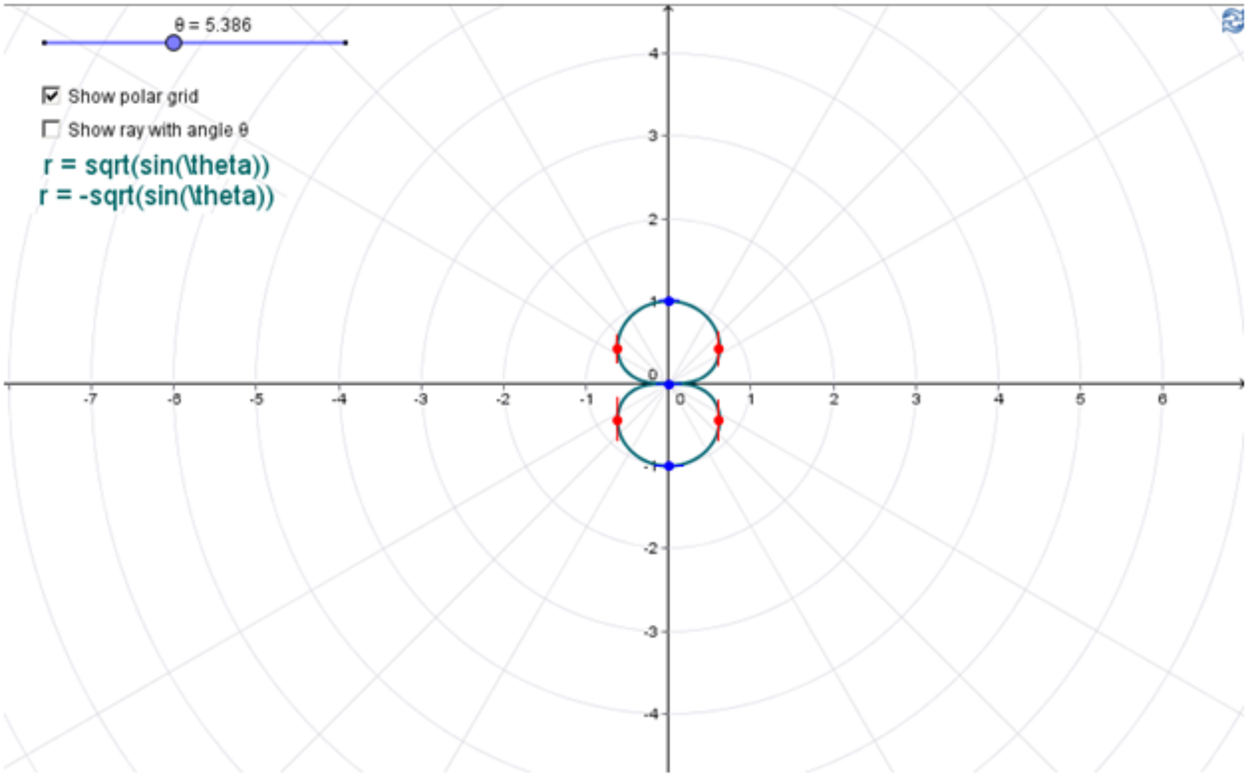
$-\frac{1}{3} = y^{\frac{4}{3}}$

In other words, for all $y = \pm\left(\frac{1}{3}\right)^{\frac{4}{3}}$ = 0.2311, -0.2311

$x = \pm\sqrt{y^{\frac{2}{3}} - y^2}$ =0.5685, - 0.5685

d. ANSWER:

Horizontal slopes at	**Vertical slopes at**
(0,-1)	**(0.2311, 0.5685)**
(0,0)	**(0.2311,- 0.5685)**
(0,1)	**(-0.2311, 0.5685)**
	(-0.2311, -0.5685)



22- Find arc length for each and arrange them from highest to lowest

a)

$$f(x)=\frac{2}{3}x^{\frac{2}{3}} \quad ; \quad 1\le x\le 8$$

$$f'(x)=\frac{4}{9}x^{\frac{-1}{3}}$$

$$L=\int_1^8\sqrt{1+(\frac{4}{9}x^{\frac{-1}{3}})^2}\,dx \rightarrow L=\int_1^8\sqrt{1+\frac{16}{81x^{\frac{2}{3}}}}\,dx \rightarrow L=\int_1^8\sqrt{\frac{81x^{\frac{2}{3}}+16}{81x^{\frac{2}{3}}}}\,dx \rightarrow L=\int_1^8\frac{1}{9x^{\frac{1}{3}}}\sqrt{81x^{\frac{2}{3}}+16}\,dx$$

$$L=\int_1^8\frac{1}{9x^{\frac{1}{3}}}\sqrt{\frac{81}{16}x^{\frac{2}{3}}+1}\,dx \rightarrow \therefore let \quad u=\frac{82}{26}x^{\frac{2}{3}} \quad ; \quad du=\frac{27}{8x^{\frac{2}{3}}}dx \rightarrow L=\int_{\frac{81}{16}}^{\frac{85}{4}}\frac{8}{243}\sqrt{u+1}\,du \rightarrow L=\int_{\frac{97}{16}}^{\frac{85}{4}}\frac{8}{243}\sqrt{z}\,dz$$

$$L=\frac{8}{243}\left[\frac{2}{3}z^{\frac{3}{2}}\right]_{\frac{97}{16}}^{\frac{85}{4}} \rightarrow L=\frac{16}{729}\left[\frac{85^{\frac{3}{2}}}{4}-\frac{97^{\frac{3}{2}}}{16}\right]$$

b) $r = e^{3\theta} \qquad 0 \le \theta \le 2\pi$

$r' = 3e^{3}$

$$L = \int_0^2 \sqrt{(e^{3\theta})^2 + (3e^{3\theta})^2 d\theta} \qquad L = \int_0^2 \sqrt{10e^{6\theta}} d\theta \qquad L = \sqrt{10}\int_0^2 e^{3\theta} d\theta$$

$$L = \sqrt{10}\left[\frac{1}{3}e^{3\theta}\right]_0^2 = \frac{\sqrt{10}}{3}e^{6} - \frac{\sqrt{10}}{3}$$

c) $r = 2a\sin\theta \quad ; \quad 0 \le \theta \le 2\pi \quad ; \quad r' = \frac{1}{2}\sin\theta d\theta$

$$L = \int_0^{2\pi} \sqrt{(2a\sin\theta)^2 + (2a\cos\theta)^2} d\theta \rightarrow L = \int_0^{2\pi} \sqrt{4a^2\sin^2\theta + 4a^2\cos^2\theta d\theta}$$

$$L = \int_0^{2\pi} 2ad\theta \rightarrow L = \int_0^{2\pi} 2a\theta \,\Big|_0^{2\pi} \rightarrow L = 4\pi a$$

d) $r = \sin^2(\frac{\theta}{2}) \quad ; \quad 0 \le \theta \le \pi$

therefore $r = \frac{1}{2} - \frac{1}{2}\cos\theta \rightarrow r' = \frac{1}{2}\sin\theta$

$$L = \int_0^{\pi} \sqrt{(\frac{1}{2} - \frac{1}{2}\cos\theta)^2 + (\frac{1}{2}\sin\theta)^2} d\theta \rightarrow L = \int_0^{\pi} \sqrt{\frac{1}{4} - \frac{1}{2}\cos\theta + \frac{1}{4}\cos^2\theta + \frac{1}{4}\sin^2\theta d\theta}$$

$$L = \int_0^{\pi} \sqrt{\frac{1}{2} - \frac{1}{2}\cos\theta d\theta} \rightarrow L = \int_0^{\pi} \sqrt{\sin^2(\frac{\theta}{2})d\theta} \rightarrow L = \int_0^{\pi} \sin(\frac{\theta}{2})d\theta$$

$$L = 2[-\cos(\frac{\theta}{2})]\,\Big|_0^{\pi} \rightarrow L = 2$$

e)

$$\begin{cases} x(t) = 2t \\ y(t) = t^2 - 1 \end{cases} \qquad -1 \le t \le 2 \qquad \begin{cases} x'(t) = 2 \\ y'(t) = 2t \end{cases}$$

$$L = \int_{-1}^{2} \sqrt{2^2 + (2t)^2}\, dt \qquad L = \int_{-1}^{2} \sqrt{4 + 4t^2}\, dt \qquad L = 2\int_{-1}^{2} \sqrt{1 + t^2}\, dt$$

Let $t = \tan(x)$, $dt = \sec^2(x)dx$ and solve as indefinite integral

$L = 2\int \sqrt{\sec^2(x)}\sec^2(x)dx \qquad L = 2\int \sec^3(x)\,dx$ by parts:

$u = \sec(x) \qquad v = \tan(x)$

$du = \sec(x)\tan(x)dx \qquad dv = \sec^2(x)dx$

$$L = \frac{2}{3}\sec(\theta)\tan(\theta) + \frac{2}{3}\ln\left|\sec(\theta)+\tan(\theta)\right|$$

return back to "t" notation and evaluate over limits

$$L = \frac{2}{3}\left[t\sqrt{t^2+1}\right]_{-1}^{2} + \frac{2}{3}\ln\left|\sqrt{t^2+1}+t\right|_{-1}^{2} \qquad L = \frac{2}{3}\left[2\sqrt{5}+\sqrt{2}\right] + \frac{2}{3}\ln\left|\frac{\sqrt{5}+2}{\sqrt{2}-1}\right|$$

f) $\begin{cases} x(t) = t\sin t \\ y(t) = t\cos t \end{cases} \qquad 0 \le t \le \pi \qquad \begin{cases} x'(t) = t\cos t + \sin t \\ y'(t) = -t\sin t + \cos t \end{cases}$

$L = \int_0^{\pi} \sqrt{(t\cos t + \sin t)^2 + (-t\sin t + \cos t)^2}\,dt$ expand and simplify to: $L = \int_0^{\pi} \sqrt{t^2+1}\,dt$

let t = tanx, dt = $\sec^2$xdx and solve as indefinite integral

$L = \int \sqrt{\sec^2 x}\sec^2 x\,dx$ by parts:

$u = \sec x \qquad v = \tan x$

$du = \sec x \tan x dx \qquad dv = \sec^2 x dx$

$$L = \frac{1}{2}\sec\theta\tan\theta + \frac{1}{2}\ln|\sec\theta + \tan\theta|$$

return back to "t" notation and evaluate over limits

$$L = \frac{1}{2}\left[\frac{t}{\sqrt{t^2+1}}\right]\Big|_0^{\pi} + \frac{1}{2}\ln\left|\frac{1}{\sqrt{t^2+1}} + t\right|\Big|_0^{\pi} \qquad L = \frac{1}{2}\left[\frac{\pi}{\sqrt{\pi^2+1}}\right] + \frac{1}{2}\ln\left|\frac{1}{\sqrt{\pi^2+1}} + \pi\right|$$

Highest to lowest (let a = 1 for part c):

b > a > c > e > d > f

23- Find the Surface area rotated about given axis

a) $f(x) = \sqrt{x} - \frac{1}{3}x^{\frac{3}{2}} \qquad 1 \le x \le 3 \qquad x-axis$

$$\frac{d}{dx}\left(\sqrt{x}-\frac{x^{3/2}}{3}\right)=\frac{1}{2\sqrt{x}}-\frac{1}{2}\sqrt{x}$$

$$SA=2\pi\int_1^3\left[\sqrt{x}-\frac{\left(x^{3/2}\right)}{3}\right]\sqrt{1+\left(\frac{1}{2\sqrt{x}}-\frac{1}{2}\sqrt{x}\right)^2}\rightarrow 2\pi\int_1^3\left[\sqrt{x}-\frac{\left(x^{3/2}\right)}{3}\right]\sqrt{1+\frac{(x-1)^2}{4x}}\rightarrow 2\pi\int_1^3\left[\sqrt{x}-\frac{\left(x^{3/2}\right)}{3}\right]\cdot\frac{(x+1)}{2\sqrt{x}}$$

$$SA=2\pi\int_1^3\frac{(x+1)}{2}-\frac{x^{5/2}+x^{3/2}}{6\sqrt{x}}\rightarrow 2\pi\left[\int_1^3\frac{x}{2}+\int_1^3\frac{1}{2}-\int_1^3\frac{x^2}{6}+\int_1^3\frac{x}{6}\right]\rightarrow 2\pi\left[\frac{x^2}{4}+\frac{x}{2}-\frac{x^3}{18}+\frac{x^2}{12}\right]_1^3\rightarrow 3-\frac{7}{9}=\frac{40\pi}{9}$$

b) $r=e^{3\theta}\qquad 0\le\theta\le 2$

$$SA=2\pi\int_0^{\frac{\pi}{2}}\left[e^{3\theta}\cos\theta\right]\sqrt{\left(e^{3\theta}\right)^2+\left(3e^{3\theta}\right)^2}\,d\theta\rightarrow 2\pi\int_0^{\frac{\pi}{2}}\left[e^{3\theta}\cos\theta\right]\sqrt{10}e^{3\theta}d\theta\rightarrow 2\sqrt{10}\pi\int_0^{\frac{\pi}{2}}\left[e^{6\theta}\cos\theta\right]d\theta$$

$$\therefore\rightarrow\begin{cases}u=e^{6\theta} & dv=\cos\theta d\theta\\ du=6e^{6\theta}d\theta & v=\sin\theta\end{cases}\rightarrow 2\sqrt{10}\pi\left[e^{6\theta}\sin\theta-6\int_0^{\frac{\pi}{2}}\left[e^{6\theta}\sin\theta\right]d\theta\right]\rightarrow 2\sqrt{10}\pi\left[\frac{-e^{6\theta}\sin\theta+6e^{6\theta}\cos\theta}{5}\right]_0^{\pi/2}$$

$$\rightarrow\frac{-2\sqrt{10}\pi e^{3\pi}}{5}$$

c) $r=2a\cos\theta\qquad 0\le\theta\le\frac{\pi}{2}\qquad Polar-axis$

$$SA=2\pi\int_0^{\frac{\pi}{2}}2a\cos^2\theta\cdot\sqrt{(2a\cos\theta)^2+(-2a\sin\theta)^2}\,d\theta\rightarrow 2\pi\int_0^{\frac{\pi}{2}}4a^2\cos^2\theta d\theta\rightarrow 8a^2\pi\int_0^{\frac{\pi}{2}}\frac{1+\cos 2\theta}{2}d\theta$$

$$SA=8a^2\pi\left[\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta\right]_0^{\pi/2}=2a^2\pi^2$$

d) $\begin{cases}x(t)=\sec t\\ y(t)=\tan t\qquad 0\le\theta\le\frac{\pi}{2}\end{cases}\qquad y-axis$

$$SA=2\pi\int_0^{\frac{\pi}{2}}\sec t\sqrt{\sec^2 t\tan^2 t+\sec^4 t}\,dt\rightarrow 2\pi\int_0^{\frac{\pi}{2}}\sec t\sqrt{\frac{\sin^2 t}{\cos^2 t}+\frac{1}{\cos^2 t}}dt\rightarrow 2\pi\int_0^{\frac{\pi}{2}}\sec^2 t\sqrt{\sin^2 t+1}dt\rightarrow\therefore let\quad \sin t=$$

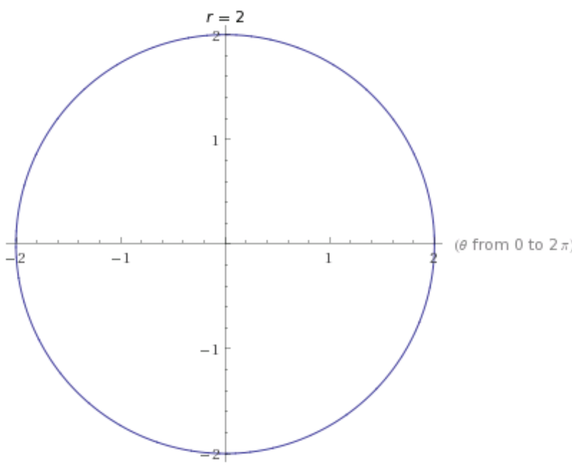
$$SA=2\pi\int_0^{\frac{\pi}{2}}\sec^3 t dt\rightarrow 2\pi\left[\sec t\tan t-\int_0^{\frac{\pi}{2}}\sec t\tan^2 t dt\right]\rightarrow 2\pi\left[\frac{1}{2}\sec t\tan t-\frac{1}{2}\ln\left|\sec t+\tan t\right|\right]_0^{\pi/2}\rightarrow 2\pi\left[\frac{1}{2}\right]$$

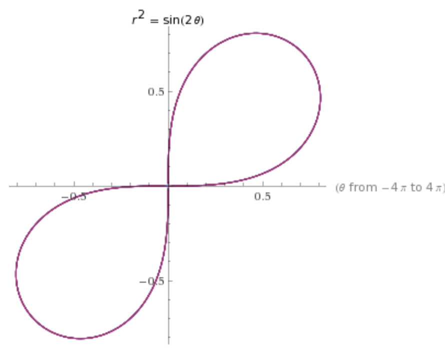
e) $\begin{cases} x(t) = 4\cos t \\ y(t) = 4\sin t \qquad 0 \le t \le \dfrac{\pi}{2} \qquad x-axis \end{cases}$

$$SA = 2\pi\int_0^{\frac{\pi}{2}} 4\sin t\sqrt{(-4\sin t)^2 + (4\cos t)^2}\,dt \to 32\pi\int_0^{\frac{\pi}{2}} \sin t dt \to \left[-32\pi\cos t\right]_0^{\pi/2} = 32\pi$$

f) $f(x) = \dfrac{1}{3}x^3 \qquad 0 \le x \le 2 \qquad y-aixs$

24-Find the area outside of $r^2 = Sin2\theta$ and inside of $r = 2$ (DO NOT USE GEOMETRY)





$$\frac{1}{2}\int_0^{2\pi} 4d\theta - \frac{1}{2}\int_0^{\pi} \sin^2(2\theta)d\theta = \left[4\pi - \frac{1}{2}\theta^2\cos 2\theta\right]_0^{\pi} = \left(4\pi - \frac{\pi^2}{2}\right)$$

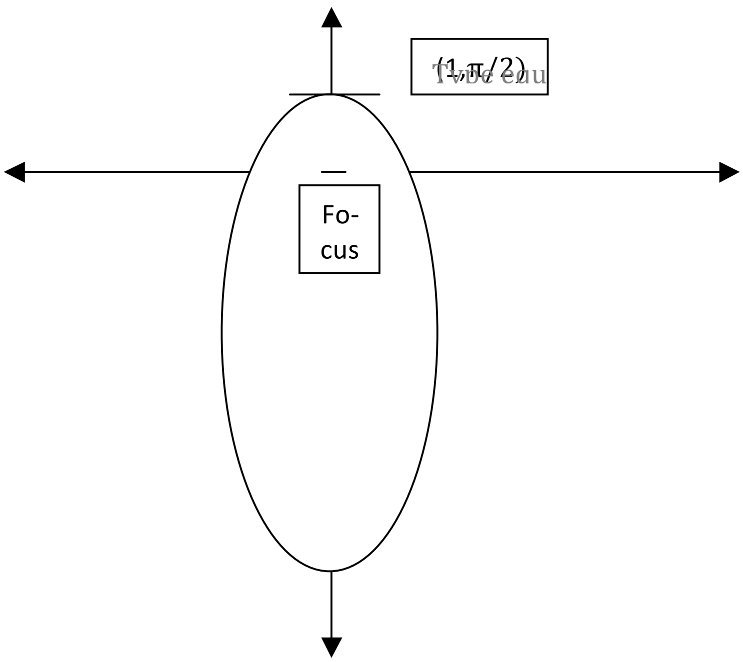
25- Use polar coordinate to find the area below line $x + y = 2$ from 0 to 2.

$$Polar : r(\sin\theta + \cos\theta) = 2r = \frac{2}{\sin\theta + \cos\theta}\; A = \int \frac{1}{2}(r\theta)^2\,d\theta A = \frac{1}{2}\int \frac{4}{1+\sin(2\theta)\theta}d\theta = 2\int \frac{d\theta}{1+\sin(2\theta)}$$

$$= 2\int \frac{d\theta}{1+\sin(2\theta)} \cdot \frac{1-\sin(2\theta)}{1-\sin(2\theta)} = 2\int \frac{1-\sin(2\theta)}{\cos^2(2\theta)}d\theta = 2\left(\int \frac{1}{\cos^2(2\theta)}d\theta - \int \frac{\sin(2\theta)}{\cos^2(2\theta)}d\theta\right)$$

$$= 2\left(\int \sec^2(2\theta)d\theta - \frac{1}{2\cos(2\theta)}\right) = 2\frac{\tan(2\theta)}{2} - \frac{1}{\cos(2\theta)} = \left(\tan\left(2\frac{\pi}{2}\right) - \frac{1}{\cos(2\frac{\pi}{2})}\right) - \left(\tan(0) - \frac{1}{\cos(0)}\right)$$

26- Write the polar and rectangular equation of a conic with focus at the origin and $e = 0.8$, vertex $(1, \frac{\pi}{2})$ then graph it.



Eccentricity is defined as the ratio of half the distance between its two foci, to the length of the Semi-major axis.

E = c/a therefore E = c/1+c

0.8(1+c) = c 0.8+0.8c = c 0.8 = 0.2c

c = 4, therefore 1+c is a, thus a = 5(Major Axis)

Due to the definition of an ellipse, $a^2 = b^2 + c^2$, where b is minor axis

$4^2 + b^2 = 5^2$ $b^2 = 25 - 16$ **b = 3**

$\frac{x^2}{b^2} + \frac{(y-k)^2}{a^2} = 1$ $\frac{x^2}{9} + \frac{(y-4)^2}{25} = 1$ **, This is the rectangular Form**

$$\frac{(r\cos\alpha)^2}{9} + \frac{(r\sin\alpha - 4)^2}{25} = 1$$

$$25 - 25r^2\sin^2\alpha + 9r^2\sin^2\alpha - 72r\sin\alpha + 144 = 225$$

$16r^2 + 72rsin\alpha + 56 = 0$ $8r(rsin^2\alpha + 9sin\alpha) = -7$

$r sin^2\alpha + 9 sin\alpha = -7/8r$, This is Polar Form

27- Find the arc length of the graph $x(t) = e^t - t \quad y(t) = 4e^{\frac{t}{2}}$ and $0 \le t \le \ln 2$.

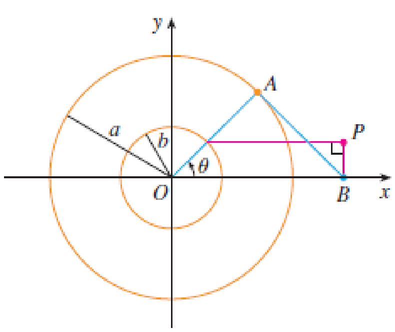
$$\frac{dx}{dt} = e^t - 1 \quad ; \quad \frac{dy}{dt} = 2e^{t/2}$$

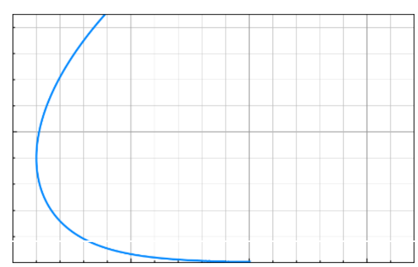
$$S = \int_0^{\ln 2} \sqrt{(2e^{t/2})^2 + (e^t - 1)^2}\, dt$$

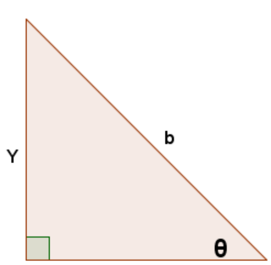
$$\rightarrow \int_0^{\ln 2} \sqrt{e^{2t} + 2e^t + 1}\, dt = \int_0^{\ln 2} e^t + 1 dt = \left[e^{\ln 2} + \ln 2\right] - \left[e^0 + 0\right] = 1 + \ln 2$$

28-

If a and b are fixed numbers, find the parametric equations for the curve that consists of all possible positions of the point P in the figure, using the angle θ as the parameter. The line segment AB is tangent to the larger circle.

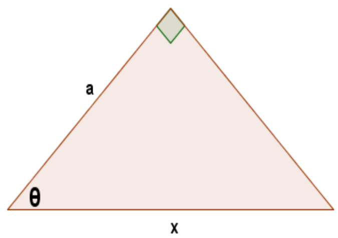






$$Sin(\theta) = \frac{y}{b}$$

$$y = b\, Sin(\theta)$$

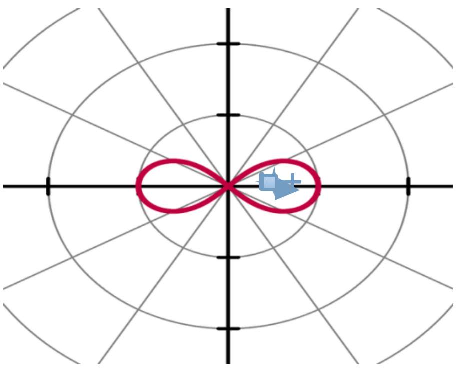


$\text{Cos}(\theta) = \frac{a}{x}$

$x = a\,\text{Sec}(\theta)$

29- Same as #27 solution with only different sign

30- Find the surface area generated by rotating the $r^2 = \cos(2\theta)$ about the polar



Surface area of Polar: $S = 2\pi \int_a^b r\cos\theta \sqrt{r^2 + r'^2}\, d\theta$

1. Find the derivative of $r = \sqrt{\cos(2\theta)}$

$$r' = \left(\sqrt{\cos(2\theta)}\right)'$$

$$r' = \frac{1}{2}(\cos(2\theta))^{-\frac{1}{2}}(-\sin(2\theta))2$$

$$r' = -\frac{\sin(2\theta)}{\sqrt{\cos(2\theta)}}$$

2.From the picture, we can see the whole area is equal to 4 times of the shade area, so the equation is

$$s = 4 * 2\pi \int_0^{\frac{\pi}{4}} \sqrt{\cos 2\theta} * \cos\theta \sqrt{\cos 2\theta + \left(\frac{-\sin 2\theta}{\sqrt{\cos 2\theta}}\right)^2}\, d\theta$$

$$s = 8\pi \int_0^{\frac{\pi}{4}} \sqrt{\cos 2\theta} * \cos\theta \sqrt{\frac{(\cos 2\theta)^2 + (\sin 2\theta)^2}{\cos 2\theta}}\, d\theta$$

$$s = 8\pi \int_0^{\frac{\pi}{4}} \sqrt{\cos 2\theta} * \cos\theta \frac{1}{\sqrt{\cos 2\theta}}\, d\theta$$

$$s = 8\pi \int_0^{\frac{\pi}{4}} \cos\theta d\theta$$

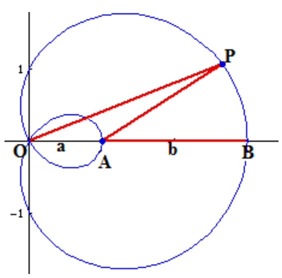
3.Then we can integrate the equation.

$$s = 8\pi * \sin\theta \Big]_0^{\frac{\pi}{4}}$$

$$s = 8\pi\left(\sin\frac{\pi}{4} - \sin 0\right)$$

$$s = 8\pi * \frac{\sqrt{2}}{2} = 4\sqrt{2}\pi \approx 17.7715$$

31- Use your pre-Calculus knowledge to find the relation between "a" and "b" such that $3\angle OPA = \angle PAB$ for any point p on the curve $r = a + bCos\theta$



$\angle AOP = \theta$

$\angle OPA = \alpha$ Show that $\angle\theta = 2\angle\alpha$

$\angle BAP = 3\alpha$

Use law of sine in ΔOPA

$\frac{Sin(180-3\alpha)}{OP} = \frac{Sin\alpha}{a}$ We know that $OP = a + bCos\theta$ and

$Sin(180-3\alpha) = Sin(3\alpha) = Sin(\theta+\alpha) = Sin\theta Cos\alpha + Sin\alpha Cos\theta$

$\frac{Sin\theta\ Cos\alpha + Sin\alpha\ Cos\theta}{a + bCos\theta} = \frac{Sin\alpha}{a}$ Cross multiply and simplify

$aSin\alpha + bCos\theta\ Sin\alpha = aSin\theta\ Cos\alpha + aSin\alpha\ Cos\theta$ Change all the θ to α

$$aSin\alpha + b(Cos^2\alpha - Sin^2\alpha)\ Sin\alpha = 2aSin\alpha\ Cos^2\alpha + aSin\alpha\ (Cos^2\alpha - Sin^2\alpha)$$

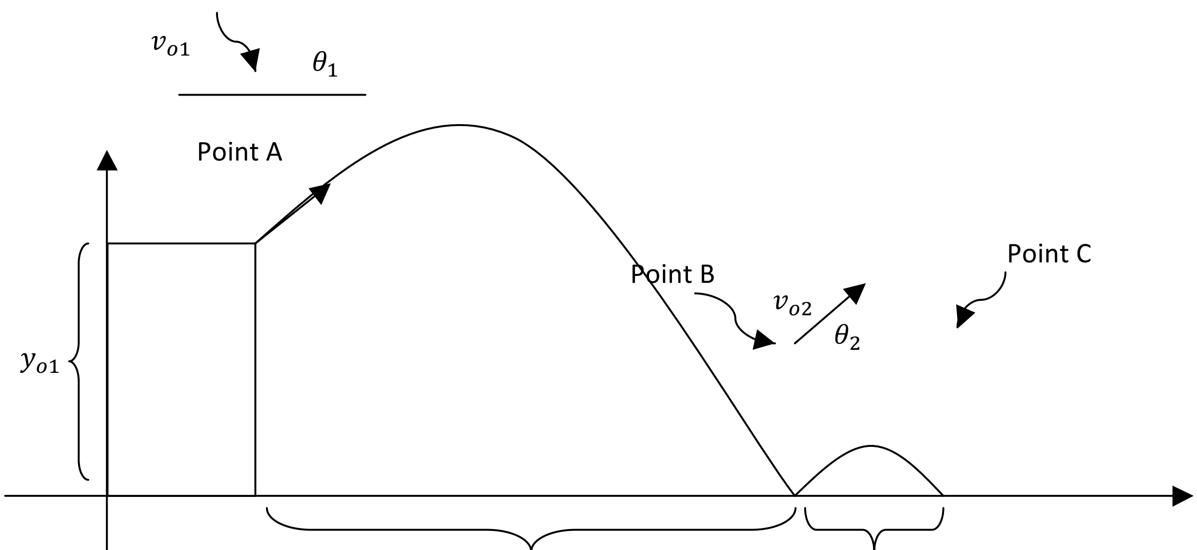
Rearrange and simplify

$(b-a)(Cos^2\alpha - Sin^2\alpha)\ Sin\alpha = aSin\alpha\ (2Cos^2\alpha - 1)$ Simplify

Identity $2Cos^2\alpha - 1 = (Cos^2\alpha - Sin^2\alpha)$

$b - a = a$ then we have $2a = b$

32- A particle launched from height of 30m with speed of 10m/s and angle of 60° in space which offers a gravitational acceleration of -10 $\frac{m}{s^2}$. As it bounces off the ground looses 2/3 of its vertical velocity. Find the distance between the first and the second time that the particle hits the ground.



Point A to Point B

X	Y
$\Delta x = (v_{oX} Cos(\theta))t$ | solve for t $t = \frac{\Delta x}{V_{oX} Cos(\theta)}$	$\Delta y = (V_{oY} Sin(\theta))t + \frac{1}{2}at^2$

Convert to Rectangular form- "by solving for "t" we eliminate the unknown and are able to plug into the second equation to get y as a function of x. Then we solve for x.

$$\Delta y = (V_{oY} Sin(\theta)t) + \frac{1}{2}at^2 = \Delta y = (V_{oY} Sin(\theta))\left(\frac{\Delta x}{V_{oX} Cos(\theta)}\right) + \frac{1}{2}a\left(\frac{\Delta x}{V_{oX} Cos(\theta)}\right)^2$$

$$\Delta y = \Delta x \tan(\theta) + \frac{a\Delta x^2}{2V_{oX}^2 (Cos(\theta))^2}$$

$$0 - 30m = \Delta x \tan(60^o) + \frac{(10\, m/s^2)\Delta x^2}{2V_{oX}^2 (Cos(\theta))^2}$$

$$-30 = 1.7\Delta x - 0.2\Delta x^2 \quad >>>> (\Delta x = x_f - x_i \quad 'x_i' in \ this \ case \ is \ 0 \ since \ we \ are \ looking \ for \ the \ final \ x \ distance \ 'x_f'$$

$$0.2x^2 - 1.7x - 30 = 0$$

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a} = \frac{-(-1.7) + \sqrt{(1.7)^2 - 4(0.2)(-30)}}{2(0.2)} = -14 \quad and \quad 22$$

$$\Delta x_2 = 22m$$

Solve for Time

$$t = \frac{x_f - x_i}{V_{OX} Cos(\theta)} = \frac{22m - 0m}{(10\, m/s^2) Cos(60^o)} = 4.4s$$

Solve for Velocity at impact

$$V_{fx} = V_{ix} Cos(\theta) = (10\, m/s) Cos(60^o) = 5\, m/s$$

$$V_{fx} = V_{ix} Sin(\theta) - gt = (10\, m/s) Sin(60^o) - (10\, m/s^2)(4.4s) = -35\, m/s$$

(traveling in negative y direction)

B to C

$$V_{oy} = 35\, m/s - (35\, m/s \cdot \frac{2}{3}) = 12\, m/s$$

$$V_{ox} = 5\, m/s$$

Solve for θ after the bounce

$$\theta = \tan^{-1}\left(\frac{V_{oy}}{V_{ox}}\right) = \tan^{-1}\left(\frac{12}{5}\right) = 67^{o}$$

Solve for Max distance-(same process that was done for A to B)

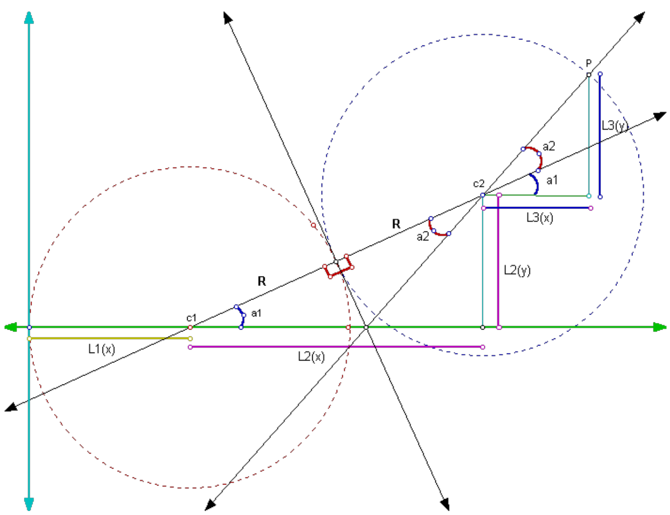
$$\Delta y = \Delta x\left(\frac{V_{oy}}{V_{ox}}\right)\tan(\theta) + \frac{a\Delta x^2}{2V_{oX}^2 (Cos(\theta))^2}$$

$$\Delta x_2 = \frac{(V_{oy})(V_{ox})(\sin(2\theta))}{g} = \frac{(12\, m/s)(5\, m/s)(\sin(2x67^0))}{10\, m/s^2} = 4.3m$$

The distance between the first and second bound is 4.3m

33- Show the equations for $Epithrochoid\begin{cases} Rose \\ Hypotrochoid \\ Cardioid \\ Nephroid \end{cases}$ and $Hypotrochoid\begin{cases} Hypocycloid \\ Ellipse \\ Deltoid \\ Astroid \end{cases}$

Cardiod



Circles c_1 and c_2 have radius R. The center of circle c_1 is located at $(R,0)$. Center c_2 is located a distance of $2R$ from center c_1.

Finding the parametric equations using what we know so far:

$$x = L_1(x) + L_2(x) + L_3(x)$$
$$y = L_2(y) + L_3(y)$$

$L_1(x)$ is equal to the radius (R). Now, form a right triangle by sending a perpendicular line from the x-axis connecting to center c_2.

The hypotenuse equals twice the radius $2R$ and the angle of the triangle is a_1.

The horizontal distance $L_2(x)$ is given by $(2R)\cos a_1$, the vertical distance $L_2(y)$ is given by $(2R)\sin a_1$.

$$x = R + \left[\left(2R\cos a_1\right) + L_3(x)\right]$$
$$y = \left[\left(2R\sin a_1\right) + L_3(y)\right]$$

To find these last two measurements $L_3(x)$ and $L_3(y)$, another right triangle is formed with the adjacent leg parallel to the x-axis and the opposite leg parallel to the y-axis.

The angle measured for this triangle is the sum of the angles a_1 and a_2.

The hypotenuse is equal to the radius of the circle (R). The horizontal distance $L_3(x)$ is given by $(R)\cos(a_1 + a_2)$, the vertical distance $L_3(y)$ is given by $(R)\sin(a_1 + a_2)$.

$$x = R + \left[\left(2R\cos a_1\right) + (R)\cos(a_1 + a_2)\right]$$
$$y = \left[\left(2R\sin a_1\right) + (R)\sin(a_1 + a_2)\right]$$

Because the arc length traveled by the two points is equal and the radius is equal, the angles a_1 and a_2 are equal. We will substitute a **'t'** in for a_1 and a_2. So the new equations are:

$$x = R + \left[\left(2R\cos t\right) + (R)\cos(2t)\right]$$
$$y = \left(2R\sin t\right) + (R)\sin(2t)$$

First pull R out. Then using the double angle formulas to simplify the equations.

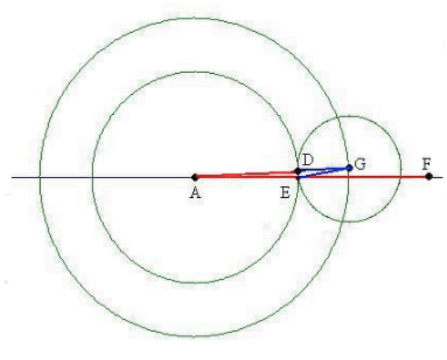
$$x = R\left[1 + 2\cos(t) + \cos(2t)\right]$$
$$y = R\left[2\sin(t) + \sin(2t)\right]$$
$$\cos(2t) = 2\cos^2(t) - 1, \quad \sin(2t) = 2\sin(t)\cos(t)$$
$$x = R\left[1 + 2\cos(t) + 2\cos^2(t) - 1\right]$$
$$y = R\left[2\sin(t) + 2\sin(t)\cos(t)\right]$$

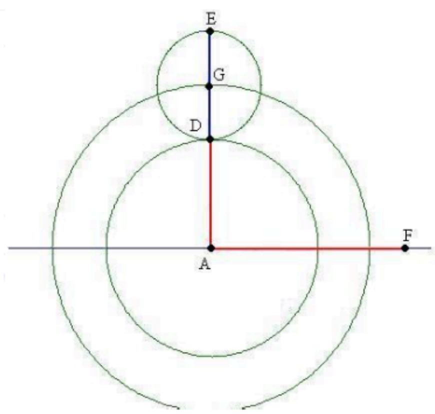
And in final parametric form: $x = 2R\cos(t)\left[1 + 2\cos(t)\right]$, $y = 2R\sin(t)\left[1 + \cos(t)\right]$

Nephroid

Deriving the formula for the Nephroid

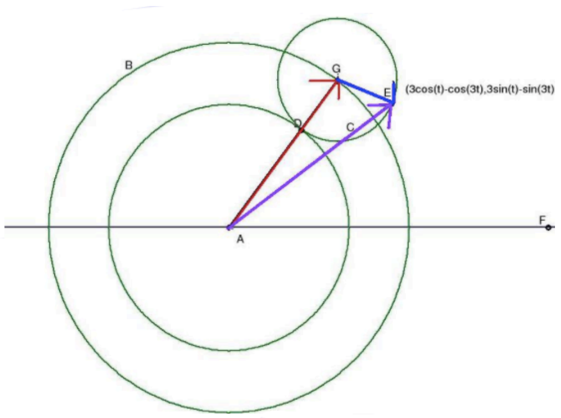
First to determine the relationship between the changes in angle from the large circle to the small circle, we analyze the relationship between point D and E as they traverse the outside of their respected circles. As point D travels $\frac{\pi}{2}$ point E travels π, therefore $\angle DGE$ (blue) is twice as large as $\angle FAD$ (red).



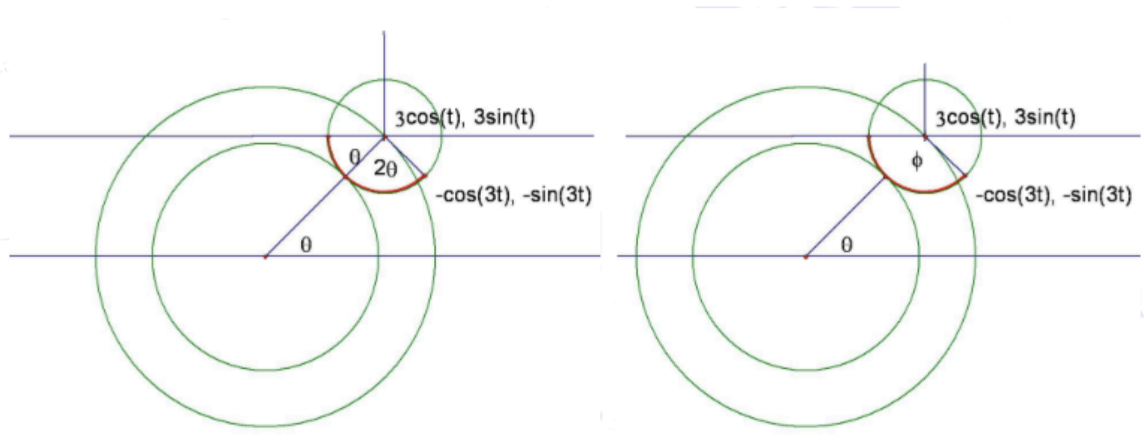


From this relationship we can start the parametrization of the nephroid. We can do this with a vector approach. First we will get to the center of circle C (red vector) then get a vector to point towards the point of interest (blue vector). Then by adding these two vectors (purple vector) we will arrive at the parametric equation for the Nephroid.

$$\overrightarrow{AG} + \overrightarrow{GE} = \overrightarrow{AE}$$



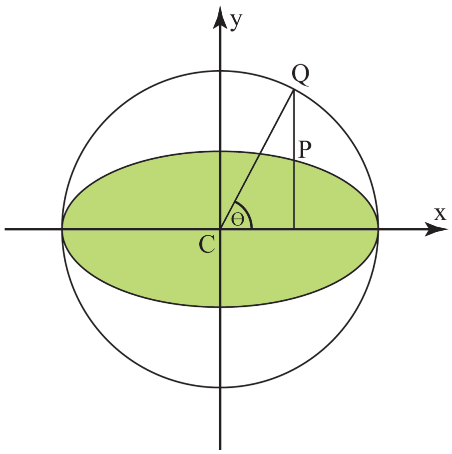
The center of circle C will always travel along a circle with a radius $3a$. Therefore, to get to this point we can use $(3a\cos\theta, 3a\sin\theta)$. Then by defining a new coordinate system we are able to locate the point of interest on the outside of the small circle. The outside of the small circle is defined by $(r\cos\theta, r\sin\theta)$. But we know that r these 2 cases equals one. We can define the angle past the second quadrant as φ. Since the point is past the second quadrant, we add π, $[\cos(\pi+\varphi), \sin(\pi+\varphi)] = [-\sin(\varphi), -\cos(\varphi)]$, and we know, by geometry, that $\varphi = 3\theta$. From this we get $[-\sin(3\theta), -\cos(3\theta)]$



$$\overrightarrow{AE} = \overrightarrow{AG} + \overrightarrow{GE}$$
$$\overrightarrow{AG} = \langle 3\cos(\theta), 3\sin(\theta) \rangle$$
$$\overrightarrow{GE} = \langle -\sin(3\theta), -\cos(3\theta) \rangle$$
$$\overrightarrow{AE} = \langle 3\cos(\theta) - \cos(3\theta), 3\sin(\theta) - \sin(3\theta) \rangle$$

Therefore, the parametric equations are: $x = [3\cos(\theta) - \cos(3\theta)]$, $y = [3\sin(\theta) - \sin(3\theta)]$

Ellipse



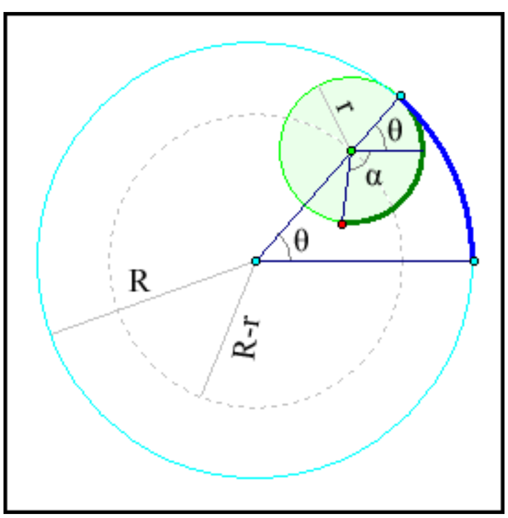
Draw a radius CQ of this circle making an angle θ with the major axis. Then CQ=a and the coordinates of Q are $(a\cos\theta \;\; and \;\; \sin\theta)$. Draw the perpendicular from Q to the major axis to meet the ellipse at P. The x-coordinate of P is also $a\cos\theta$ and since the point P lies on the ellipse:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \cos^2(\theta)+\frac{y^2}{b^2}=1$$

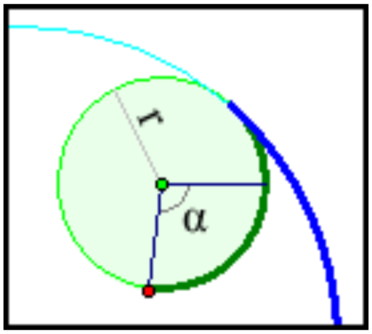
Therefore $y^2=b^2(1-\cos^2\theta)=b^2\sin^2\theta$ Thus $y=\pm b\sin\theta$

Therefore the Parametric equations of an ellipse are: $\begin{matrix} x=a\cos\theta \\ y=b\sin\theta \end{matrix}$

Hypocycloid



Let *R* be the radius of the larger circle, and let *r* be the radius of the smaller circle. For simplicity, assume the center of the larger circle is located at (0,0). Let (x_0, y_0) be the coordinates of the smaller circle's center.



This is an enlargement of the diagram above, showing the smaller circle.

From this diagram, we have:

$$x = x_0 + r\cos(\alpha)$$
$$y = y_0 - r\sin(\alpha)$$

Now let's determine the values of x_0 and y_0 in terms of θ.

Note that the center of the smaller circle lies on a circle of radius *R-r*.

From the first diagram, we have

$$x_0 = (R - r)\cos(\theta)$$
$$y_0 = (R - r)\sin(\theta)$$

Now we need to find the value of α in terms of θ.

In the diagram, the dark blue and dark green arcs must be the same length, because the smaller circle is "rolling" along the larger one. This gives us

$$\left(\frac{\theta}{2\pi}\right)(2\pi R) = \left(\frac{\alpha + \theta}{2\pi}\right)(2\pi R)$$
$$\Leftrightarrow \theta R = (\alpha + \theta)r$$
$$\Leftrightarrow \theta(R - r) = \alpha r$$
$$\Leftrightarrow \alpha = \theta\left[\frac{R - r}{r}\right]$$

So we have

$$x = x_0 + r\cos(\alpha) = (R-r)\cos(\theta) + r\cos\left[\theta\left(\frac{R-r}{r}\right)\right]$$

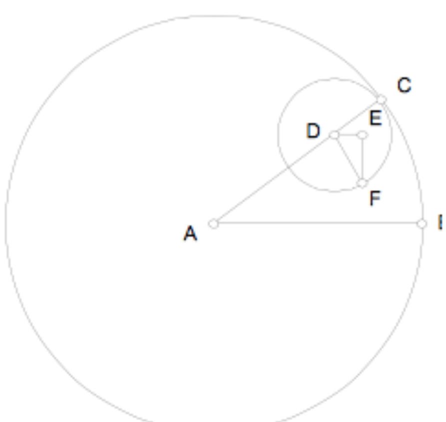
$$y = y_0 - r\sin(\alpha) = (R-r)\sin(\theta) - r\sin\left[\theta\left(\frac{R-r}{r}\right)\right]$$

Thus, we have found parametric equations for the hypocycloid:

$$x = x_0 + r\cos(\alpha) = (R-r)\cos(\theta) + r\cos\left[\theta\left(\frac{R-r}{r}\right)\right]$$
$$y = y_0 - r\sin(\alpha) = (R-r)\sin(\theta) - r\sin\left[\theta\left(\frac{R-r}{r}\right)\right]$$

Deltoid

A wheel of radius CD rolls counter-clockwise along the circumference of the larger circle. The radius of the fixed circle is either three or two-thirds as long as the wheel's radius. To find the parametric equation for the path (called the deltoid) traced by the point F, we need to find F(x,y).



Parametrization

Using the figure above, we can make the following observations.

1. Make the radius CD of the wheel be $\frac{1}{3}$ of the radius of the larger circle.

2. The length of AD will be 3 times as long as CD (written 3CD).

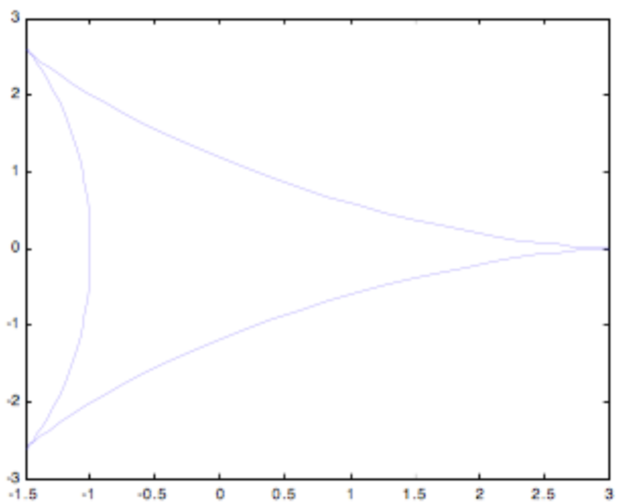
3. The line AD has length 2 CD.

4. The arc length BC will be 3CDt, and the arc length of FC is 3CDt.

5. Therefore, the coordinates of point D are $\left[2a\cos(t), 2a\sin(t)\right]$.

6. The angle between CD and DE is t.

7. Arc length of BC divided by the length CD produces the angle between AD and DF; $\frac{3at}{a} = 3t$



8. The angle 3t subtracted from the angle t gives the angle between DE and DF.

9. Angle FDE: $3t - t = 2t$.

10. Length of DE: $CD\cos(2t)$.

11. Length of EF: $CD\sin(2t)$.

12: Combining the two

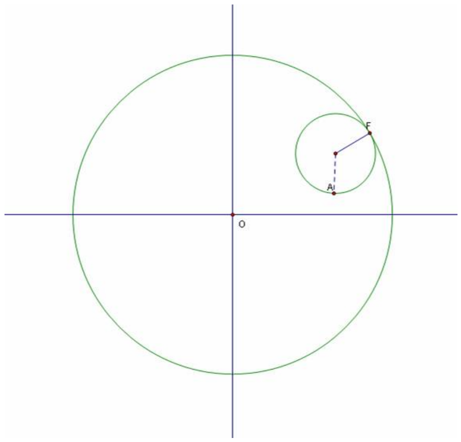
$$x = 2CD\cos(t) - CD\cos(2t)$$
$$y = 2CD\sin(t) - CD\sin(2t)$$

13. CD=a, and $t = \theta$

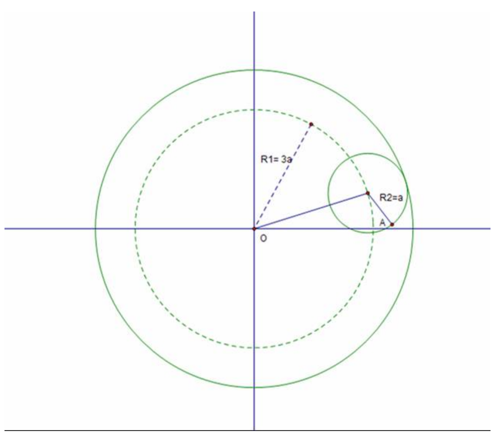
14. Replacing the variables
$$x = 2a\cos(\theta) - a\cos(2\theta)$$
$$y = 2a\sin(\theta) - a\sin(2\theta)$$

Astroid

We will now derive the parametric equations for the asteroid. We start by looking at Figure below, in this case, we have a small circle of radius a, inside a large circle of radius 4a.

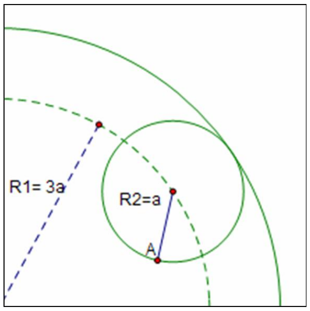


To get our equations we first notice that as the small circle moves around the large circle, it's center is always at a distance 3a from the origin. This tells that as the small circle moves it traces out another circle of radius 3a (Figure below).



Thus we note, that the curve is actually created by the combined movement of the two circles. So, we can simplify our task by finding the parametric equations of the two circles and adding them together to get the equations of the Asteroid.

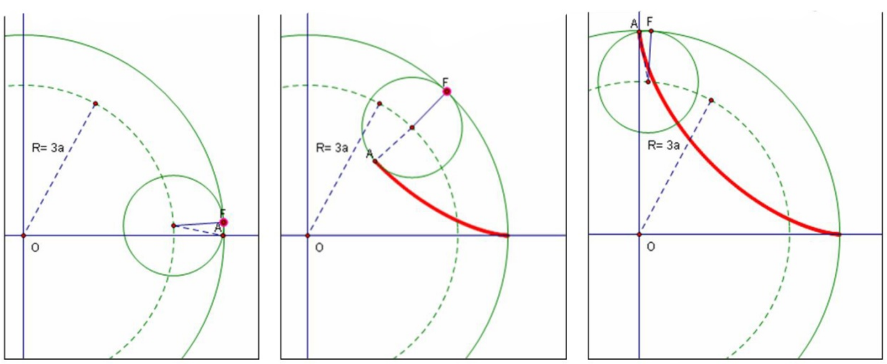
Since the parametric equations of a circle are $x = r\cos\theta, y = r\sin\theta$, and here *r* = 3*a*, we see that the parametric equations of the larger, dotted circle is $x = 3a\cos\theta, y = 3a\sin\theta$ I. Now, for the smaller circle (see Figure below).



This circle has radius a, so it must have parametric equations of, $x = a\cos\alpha, y = a\sin\alpha$. Finally we can add the 2 sets of parametric equations to get that the equations of the asteroid are:

$$x = 3a\cos\theta + a\cos\alpha$$
$$y = 3a\sin\theta + a\sin\alpha$$

However, we need to express x and y, in terms of one angle, (not two), to do that we need to find *α* in terms of *θ*. Here we look at the curve over the first quadrant of the graph (See Figure below).



During this time the first part goes through an angle of 90, the second passes through -270°, so the relationship is $\alpha = -3\theta$, so our parametric equations become.

$$x = 3a\cos\theta + a\cos(-3\theta)$$
$$y = 3a\sin\theta + a\sin(-3\theta)$$

Now we are going to simplify these parametric equations into more compact form. We start with our value of x, pull the negative out (cosine is an even function), and simplify

For the X-value equation we have,

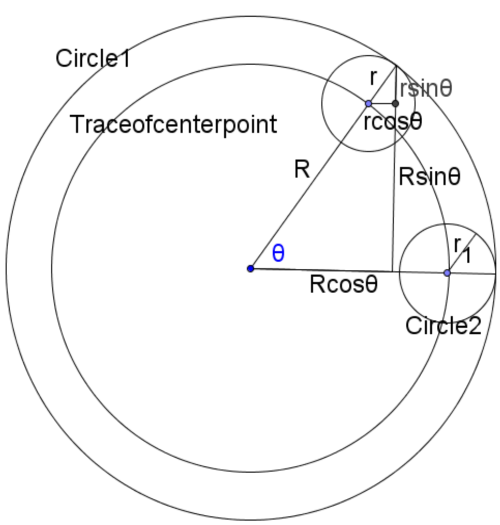
$$x = 3a\cos(\theta) + a\cos(-3\theta)$$
$$= 3a\cos(\theta) + \cos(2\theta + \theta)$$
$$= 3a\cos(\theta) + a\left[\cos(2\theta)\cos(\theta) - \sin(2\theta)\sin(\theta)\right]$$
$$= 3a\cos(\theta) + a\left[\cos^3(\theta) - \sin^2(\theta)\cos(\theta) - 2\sin^2(\theta)\cos(\theta)\right]$$

$$= a\cos(\theta)\left[3 + \cos^2(\theta) - 3\sin^2(\theta)\right]$$
$$= a\cos^3(\theta) + a\cos(\theta)\left[3 - 3\sin^2(\theta)\right]$$
$$= a\cos^3(\theta) + a\cos(\theta)\left[3\left(\cos^2(\theta) + \sin^2(\theta)\right) - 3\sin^2(\theta)\right]$$
$$= a\cos^3(\theta) + a\cos(\theta)\left[3\cos^2(\theta)\right]$$
$$= a\cos^3(\theta) + 3a\cos^3(\theta)$$
$$= 4a\cos^3(\theta).$$

Similarity we can simplify y to $y = 4a\sin^3(\theta)$ Therefore our parametric equations are:

$$x = 4a\cos^3(\theta)$$
$$y = 4a\sin^3(\theta)$$

Hypotrochoid



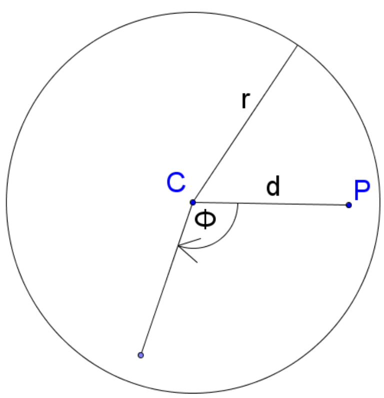
As the point *C* travels through an angle theta, it's x-coordinate is defined as $(R\cos\theta - r\cos\theta)$ and its y-coordinate is defined as $(R\sin\theta - r\sin\theta)$. The radius of the circle created by the center point is (R-r).

As the small circle goes in a circular path from zero to 2π, it travels in a counter-clockwise path around the inside of the large circle. However, the point *P* on the small circle rotates in a clockwise path around the center point.

As the center rotates through an angle theta, the point *P* rotates through an angle phi in the opposite direction.

The point *P* travels in a circular path about the center of the small circle and therefore has the parametric equations of a circle. However, since phi goes clockwise, $x = d\cos\varphi, y = -d\sin\varphi$.

Inner circle



Adding these equations to the equations for the center of the inner circle gives the parametric equations for a hypotrochoid.

$$x = R\cos\theta - r\cos\theta + d\cos\varphi$$
$$y = R\sin\theta - r\sin\theta - d\sin\varphi$$

Get φ in terms of θ

Since the inner circle rolls along the inside of the stationary circle without slipping, the arc length $r\varphi$ must be equal to the arc length $R\theta$.

$$r\varphi = R\theta$$
$$\varphi = \frac{R\theta}{r}$$

However, since the point P rotates about the circle traced by the center of the small circle, which has radius $(R-r)$, φ is equal to $\frac{(R-r)}{r}\theta$

Therefore, the equations for a hypotrochoid are

$$x = R\cos\theta - r\cos\theta + d\cos\left(\frac{(R-r)}{r}\theta\right)$$
$$y = R\sin\theta - r\sin\theta - d\sin\left(\frac{(R-r)}{r}\theta\right)$$

34- Proof a parametric equation for Prolate and Curtate

$$\sin\theta = \sin(180-\theta) = \sin(\angle APC) = \frac{AC}{a} = \frac{BD}{a} \qquad \frac{dx}{d\theta} = 1-\cos\theta$$

$$\cos\theta = -\cos(180-\theta) = -\cos(\angle APC) = -\frac{AP}{a} \qquad \frac{dy}{d\theta} = \sin\theta$$

$$x = OD - BD = a\theta - a\sin\theta \qquad y = BA + AP = a - a\cos\theta \qquad \frac{dy}{dx} = \frac{\sin\theta}{1-\cos\theta}$$

$$x = a(\theta - \sin\theta)$$
$$y = a(1-\cos\theta)$$

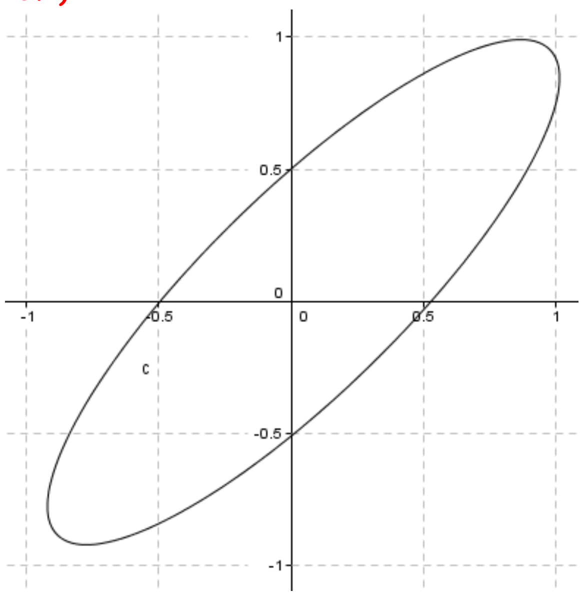
-Prolate and Curtate proof: A prolate is a cycloid with point P outside of the circle while a curtate cycloid has Point P inside of the circle. This would mean that radius a is not equal to radius b (regular cycloid). For a prolate, a is greater than b and for a curtate, a is less than b. Therefore the equation for prolate and curtate will have different radius values for a and b.

35- Given the following parametric equation $\begin{cases} x(t) = rSin(\omega t+\varphi) \\ y(t) = RSin(t) \end{cases}$ Use a graphing Calculator to graph the above equation with specific values $R = r = 1$ otherwise it is given

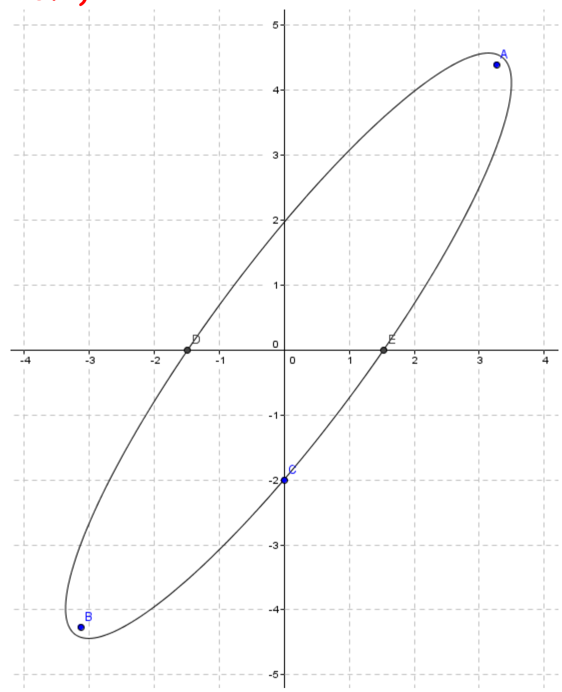
$\omega=1, \varphi=0.5$ $R=r=1$	$\omega=1, \varphi=0.5, R\neq r$	$\omega=0.5, \varphi=\pi/4$	$\omega=1, \varphi=0$
$\omega=1/2, \varphi=0$ $R=r=1$	$\omega=1/3, \varphi=0$ $R=r=1$	$\omega=1/4, \varphi=0$	$\omega=2/5, \varphi=0$

$\omega = 1/2, \varphi = 0, R = 2, r = 1$	$\omega = 3/5, \varphi = 0, R = 2, r = 1$	$\omega = 3/5, \varphi = 0, R = 2, r = 1$	$\omega = 3/5, \varphi = \pi/3$

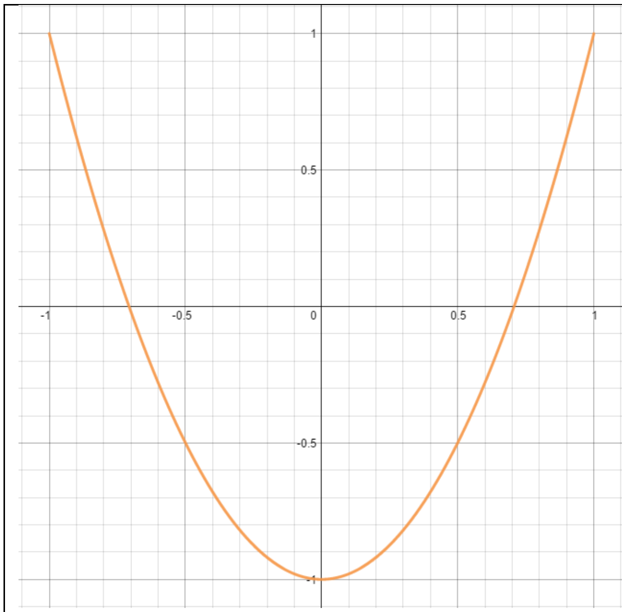
<table>
<tr>
<td>
35.1)

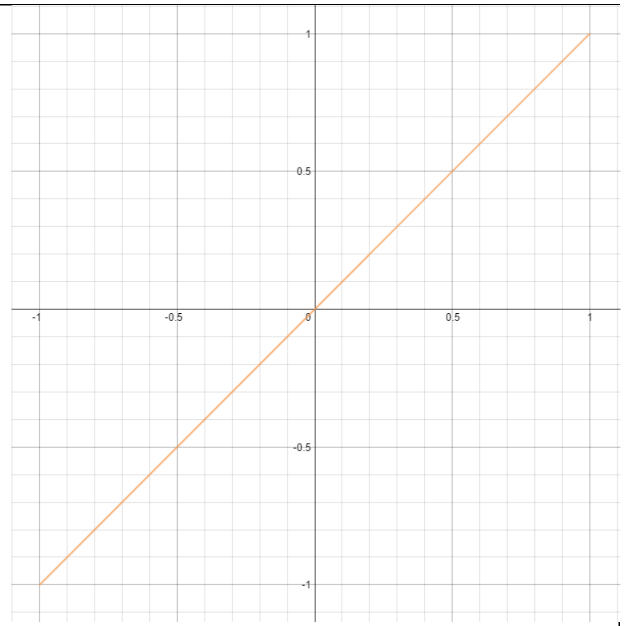


</td>
<td>
35.2)



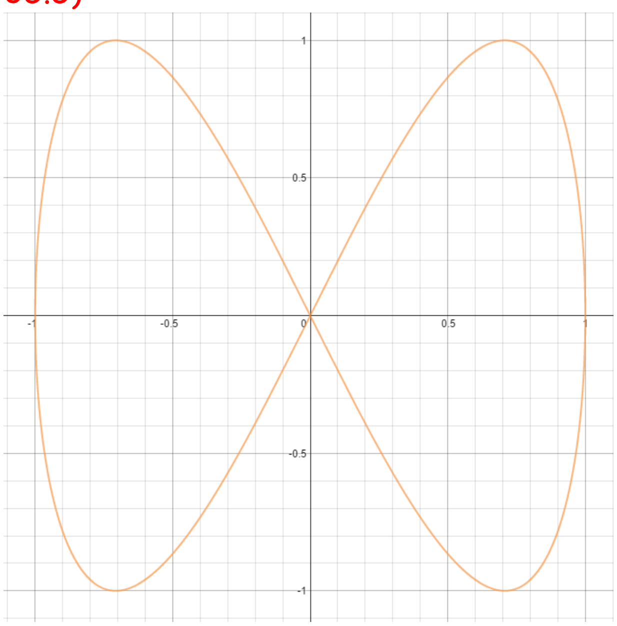
</td>
</tr>
<tr>
<td>35.3)</td>
<td>35.4)</td>
</tr>
</table>

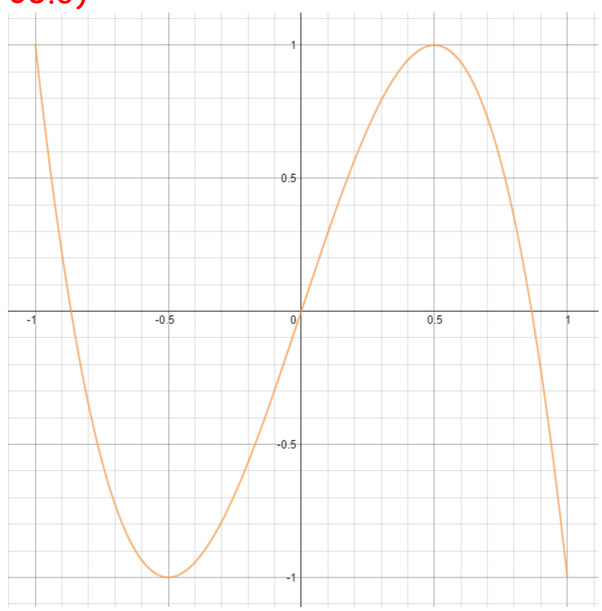




35.5)

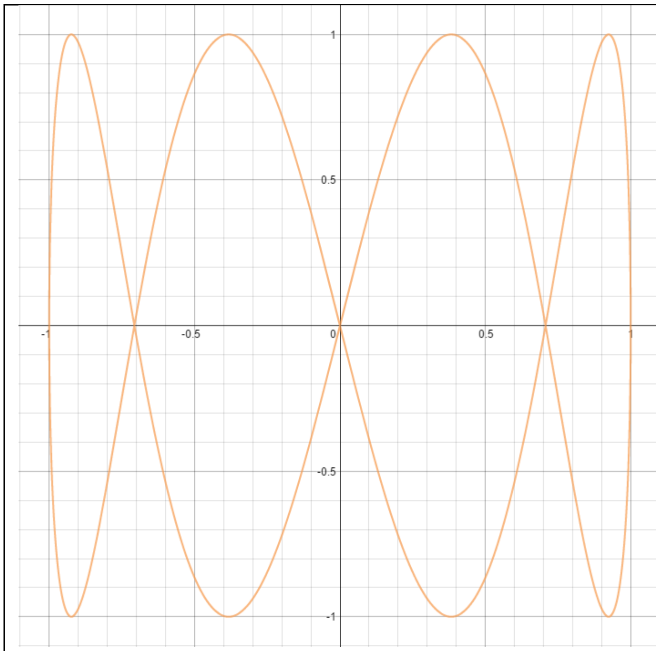
35.6)

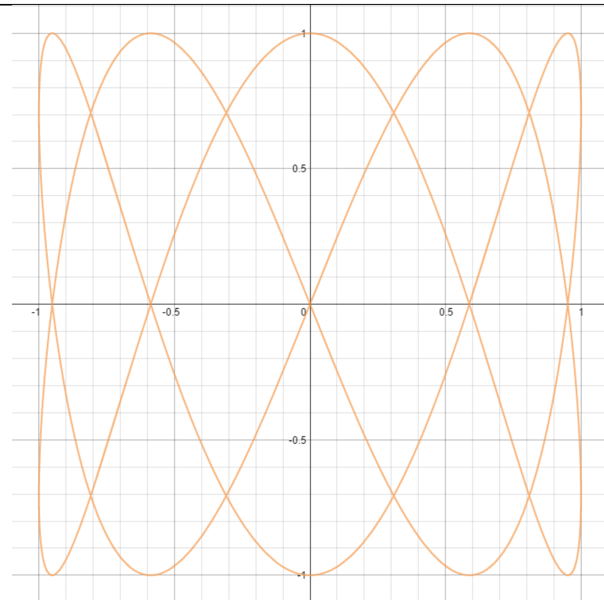




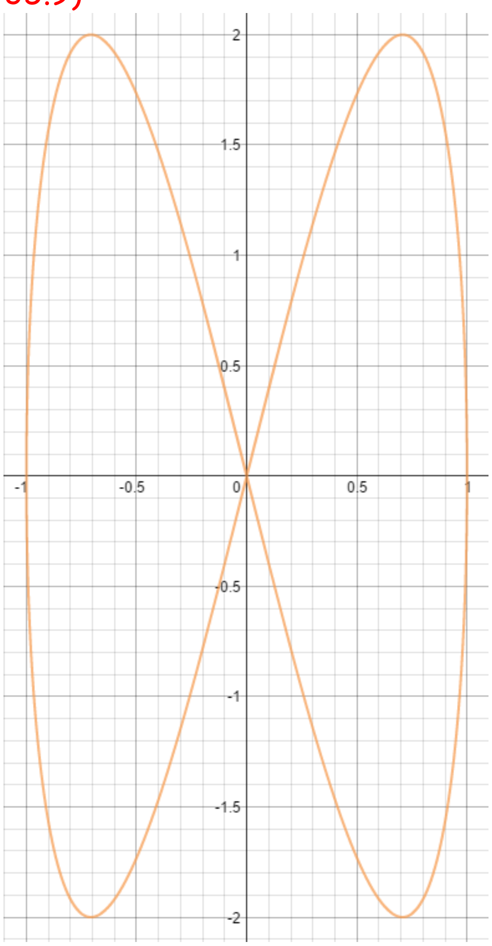
35.7)

35.8)

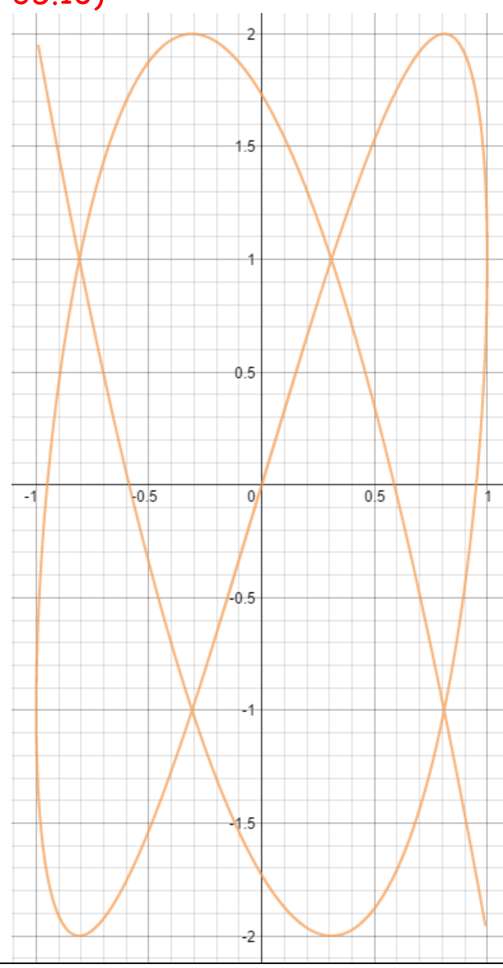

1
0.5
-0.5
-1
-1
-0.5
0
0.5
1


1
0.5
-0.5
-1
-1
-0.5
0
0.5
1

35.9)


2
1.5
1
0.5
-0.5
-1
-1.5
-2
-1
-0.5
0
0.5
1

35.10)


2
1.5
1
0.5
-0.5
-1
-1.5
-2
-1
-0.5
0
0.5
1

35.11) same as 35.10	35,12)