# Chapter 10

In this chapter we will review some of the materials from pre-calculus, Geometry and Calculus. We will look at three different coordinates (Rectangular, Parametric and Polar) plus two linear operators (Differential and integral).
Rectangular coordinate in 3D fills a three dimensional space with similar bases. ($\vec{i}, \vec{j}, \vec{k}$)
These are the standard Bases for rectangular coordinates. These standard bases have a unit magnitude and they are orthogonal to each other with unit of Length. A set of non-Standard Bases ($e_1, e_2, e_3$) are linearly independent with unit Length.

If we pick the variables as x, y, and z then any function in this space can be written as $f(x, y, z) = c \quad \forall c \in \Re$. To write this function (if possible) in explicit form, we can choose one of the variables as independent and the other two as dependent variables.

Two points in this continuum space can be connected with at least one curve $z(x, y(x)) = c$
To find the arc length of the path between these points, we have to use infinitesimal change (dx) to calculate infinitesimal length (ds) then add them up with the integration.

$$ds = \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = \sqrt{(1)^2 + (\frac{dy}{dx})^2 + (\frac{dz}{dx})^2}\ dx, \quad s = \int_{x_1}^{x_2} \sqrt{1 + (\frac{dy}{dx})^2 + (\frac{dz}{dx})^2}\ dx.$$

For case of 2D we have $s = \int_{x_1}^{x_2} \sqrt{1 + (\frac{dy}{dx})^2}\ dx$ and in case of 1D we have $s = \int_{x_1}^{x_2} \sqrt{1}\ dx$

If dimension of time is introduced to these 3D to make at 4D space and time coordinates then a few adjustment is needed.
1- All axes must have the same unit [L], so the time [S] axes must multiply by speed [L/S] to change the unit [L].
2- All the bases must be linearly independent. The 4$^{th}$ dimension of time is imaginary axes to 3D observer, and then the $ct$ axes must multiply by imaginary $i$.
3- Distance between two points must be independent of the observer.

$$ds = \sqrt{(ict)^2 + (dx)^2 + (dy)^2 + (dz)^2} = \sqrt{-1 + (\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}\ dt$$, Where C = 1 unit

$$s = \int_{x_1}^{x_2} \sqrt{-1 + \dot{x}^2 + \dot{y}^2 + \dot{z}^2}\ dt.$$

If dimension of time is introduced to the 3D Rectangular function to make at 3D parametric function then no adjustment is needed. So parametric equations are same as Rectangular equations which all the three variables are function of time $x(t), y(t), z(t)$

$$ds = \sqrt{(dx_{(t)})^2 + (dy_{(t)})^2 + (dz_{(t)})^2} = \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2 + (\frac{dz}{dt})^2}\ dt\ ,\ s = \int_{x_1}^{x_2} \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}\ dt\ .$$

This arc length formula can be reduced to $s = \int_{x_1}^{x_2} \sqrt{\dot{x}^2 + \dot{y}^2}\ dt$ and $s = \int_{x_1}^{x_2} |\dot{x}|\ dt = \int_{x_1}^{x_2} (Speed)\ dt$

You have seen some of 2D parametric equations such as Circles, ellipses, Hyperbolas, parabolas and lines.

To convert parametric to Rectangular with three steps:

1- Solve "t" in terms of x in x(t) to change it into t(x).

2- Replace t(x) with all the "t's" in y(t)

3- Simplify the equation of y(x)

$$\begin{cases} x = 3t + 2 \\ y = t^2 - 1 \end{cases}$$

1- $t = \frac{x-2}{3}$

2- $y = (\frac{x-2}{3})^2 - 1$

3- $y = \frac{1}{9}(x-2)^2 - 1$

To convert Rectangular to parametric with three steps:

1- Solve one variable (Y) in terms of the other variable (x).

2- Replace "x" with some suitable function of x(t).

3- Replace x(t) with all the "x's" in y(x).

$$y - 3 = 2Sin^2 x$$

1- $y = 2Sin^2 x + 3$

2- $t = Sinx$

3- $y = 2t^2 + 3$

4- $\begin{cases} x(t) = Sin^{-1} t \\ y(t) = 2t^2 + 3 \end{cases}$

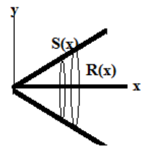
We have evaluated the area under a curve $y(x) \geq 0$ with the integration $A = \int_{x_1}^{x_2} y(x) dx$. Now

We can convert this integrant to a parametric form with $y(x) = y(x(t)) = y(t)$ and

$dx = dx(t) = \frac{dx}{dt} dt = \dot{x}(t) dt$ then the Area is $A = \int_{t_1}^{t_2} y(t)\dot{x}(t) dt$

The area of surface of rotation is $SA = \int_a^b 2\pi (Radius\ of\ rotation)(Arclength)d(variable)$

Rotation around <u>x-axes</u> $SA = \int_{x_1}^{x_2} 2\pi y(x)\sqrt{1+(y')^2}\,dx$




<u>y-axes</u> $SA = \int_{x_1}^{x_2} 2\pi\ x\sqrt{1+(y')^2}\,dx$

In parametric form: <u>x-axes</u> $SA = \int_{t_1}^{t_2} 2\pi y(t)\sqrt{\dot{x}^2+\dot{y}^2}\,dt$ and <u>y-axes</u> $SA = \int_{t_1}^{t_2} 2\pi\ x(t)\sqrt{\dot{x}^2+\dot{y}^2}\,dt$

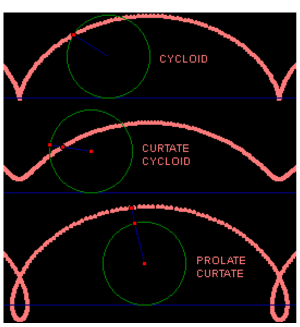
Parametric equations such as Rose and Cardiac (discussed in pre-Cal courses), now we are looking at more general cases such as Cycloid, Epicycles and Hypocycloids.

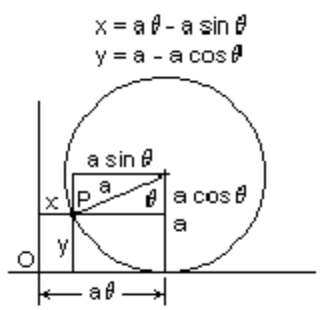
Given a wheel with radius r rolls on a flat surface, a point P a distance h from the center of the rotating wheel trace a curve called Cycloid (Circle-like).

The parametric equation of trace of point P as a function of rotated angle is:

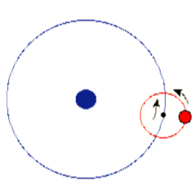
$\begin{cases} x(t) = rt - hS\text{int} \\ y(t) = r - hCost \end{cases}$ Where $\theta = \omega t$ and $\omega = 1$ Cycloid: (Circle-like) if $h = r$

<u>Prolate:</u> (Lengthened radially) If $h > r$ and <u>Curtate:</u> (Shortened radially) If $h < r$

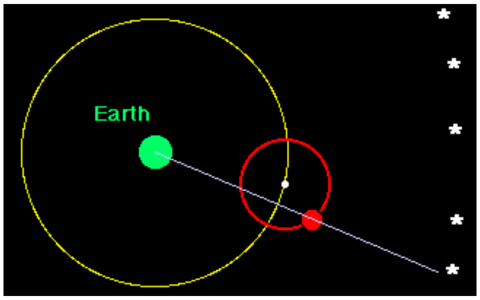


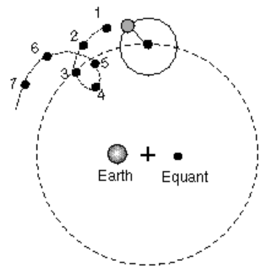




1- When center of a circle with radius “a” rotates on a surface of another circle with radius “b” without slipping, the possibility of the path is infinite. It depends to the ratio of the radii.



Number of times that the red circle spin until it goes back to initial position also depends on the radii of the circles. These functions were introduced to explain the motion of planets to be on a perfect circle and all planets (known at that time) to orbit around the earth.




As the red planet moves in the same direction of center of its orbit, the observer on Earth sees that the planet is rotating C.C.W and slows down to the point that the red planet moves against the motion of center of its orbit. Then the observer sees the planet is moving C.W. These models were constructed to make sure that the observation verifies that Earth is the center of universe.

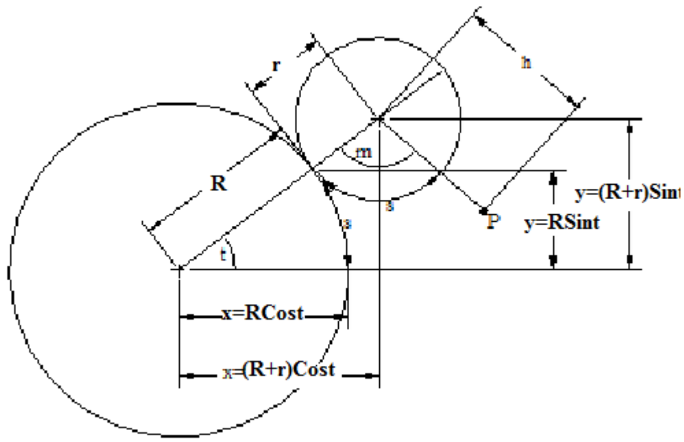
2- When a circle with radius "a" rotates tangentially inside or outside of another circle with radius "b" without slipping, the possibility of the path is infinite. It depends to the ratio of the radii.

Now, we can look at Troch-oids (Wheel-like) and with the help of a Computer or graphing Calculators, you can graph a few of these parametric functions.

Given a wheel with radius r rolls on top of a fixed wheel with radius R, a point P a distance h from the center of the rotating wheel trace a curve called Epi-troch-oid (on top-wheel-like). The parametric equation of trace of point P as a function of rotated angle is:

$$\begin{cases} x(t) = (R+r)Cos(t) - hCos((\frac{R+r}{r})t) \\ y(t) = (R+r)Sin(t) - hSin((\frac{R+r}{r})t) \end{cases} \quad \text{Where } \theta = \omega t \text{ and } \omega = 1$$

Epicycloids: The parametric equation is when $h = r$ Epitrochoid generates Epicycloids




If the rotating wheel is inside of the other wheel then the equation for parameterize t is

$$\begin{cases} x(t) = (R-r)Cos(t) + hCos((\frac{R-r}{r})t) \\ y(t) = (R-r)Sin(t) - hSin((\frac{R-r}{r})t) \end{cases}$$ and the graph is called Hypo-Troch-oid

$$Epithrochoid \begin{cases} Rose \\ Hypotrochoid \\ Cardioid \\ Nephroid \end{cases} \qquad Hypotrochoid \begin{cases} Hypocycloid \\ Ellipse \\ Detoid \\ Astroid \end{cases}$$

Rose: The parametric function is $\begin{cases} x(t) = Cos(k\theta)Sin(t) \\ y(t) = Cos(k\theta)Cos(t) \end{cases}$ and polar function is $r = Cos(k\theta)$

The parametric function is $\begin{cases} x(t) = Sin(k\theta)Sin(t) \\ y(t) = Sin(k\theta)Cos(t) \end{cases}$ and polar function is $r = Sin(k\theta)$

Cardioid: (heart- like) When $h = r = R$ $\begin{cases} x(t) = (2r)Cos(t) - rCos(2t) \\ y(t) = (2r)Sin(t) - rSin(2t) \end{cases}$

Nephroid: (kidney- Like) When $h = R = 2r$ $\begin{cases} x(t) = (3r)Cos(t) \pm 2rCos(3t) \\ y(t) = (3r)Sin(t) \pm 2rSin(3t) \end{cases}$

The plus sign when it opens up in x direction and minus sign when it opens up in y direction

Hypocycloids: The parametric equation is when $h = r$ Hypotrochoid generates hypocycloids

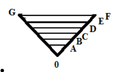
$$\begin{cases} x(t) = (R-r)Cos(t) + rCos((\frac{R-r}{r})t) \\ y(t) = (R-r)Sin(t) - rSin((\frac{R-r}{r})t) \end{cases}$$ and it can generate the following graphs when $h \neq r$

Ellipse: When $R = 2r$ Deltoid: (Delta- like) When $R = 3r$ Astroid: (Star- like) When $R = 4r$

We have seen 2D polar coordinate in pre-Calculus course. We discovered a few differences between polar and rectangular coordinates.

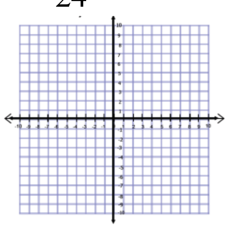
1- Polar coordinate has only one pole and infinity.

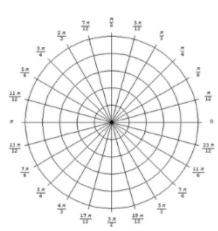
2-The bases don't have the same unit.

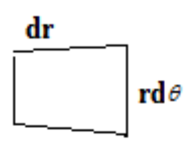
Cut out a triangular reign from Rectangular coordinate (include all the horizontal lines) with $90-\varepsilon \leq \theta \leq 90+\varepsilon$ center at origin.  Now keep origin as the pivot point, in the same plane (2D) rotate line OF and connected to line OG. All the horizontal lines will be circles. Of course to stay in 2D and perform such an operation the space must be stretchable. A line away from pivot point stretches more than the closer line. It shows that topologically speaking polar and rectangular coordinates are isomorphic to each other. And any point (function) in rectangular coordinate can be mapped on to polar coordinate and vice versa. Any point in 2D polar coordinate is represented with $(\theta\,[Rad], r\,[L])$. The bases don't have the same units. Take any cell $(dA = dxdy)$ in rectangular coordinate and compare it to any other cell. They are identical. Rectangular coordinate is isotropic and also invariant of any translation but not rotation.

In polar coordinate the arc length of all concentric circles in interval $0 \leq \theta \leq \pi/12$ are all different, as the circle radially goes outward, the arc length increases linearly.

$S_1 = \frac{2\pi R_1}{24} = \pi R_1/12 = R_1\theta$








If the angle and separation between the circles are very small then we can assume that each cell in polar coordinate is a rectangle then the area is $dA = rdrd\theta$ (the coefficient r is called the area correction). Now the area has the unit of $[L]^2$. Polar coordinate is isotropic and also invariant of any rotation but not translation.

Graph the following functions.

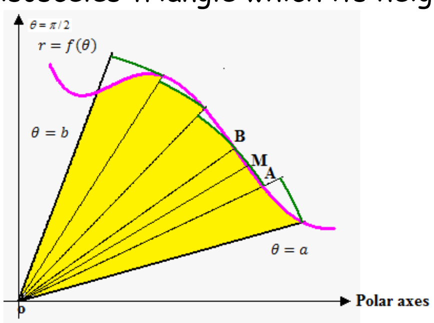
$y = x^2$ *and* $y = (x-2)^2 + 1$

$r_1 = Sin(2\theta\ )\ and\ r_2 = Sin(2\theta\ ) + 1\ and\ r_3 = Sin(2\theta\ + \pi)$

The transformation from one coordinate to another is related to the area correction

$$\begin{cases} x = rCos\theta \\ y = rSin\theta \end{cases} \quad and \quad \begin{cases} r = \sqrt{x^2 + y^2} \\ \theta = Tan^{-1}(\frac{y}{x}) \end{cases}$$

Pay attention that $\theta$ is unit-less and independent of r. So, r is a function of $\theta$ or $r = f(\theta)$.
Let's develop some formulas in polar coordinate.
Take a sector with infinitesimal change in angle (OAB). It can be approximate as a isosceles triangle which its height (OM) is equal to its two equal sides. (OM=OA=OB)




Area of $\Delta OAB = \frac{1}{2}(OM)(AB) = \frac{1}{2}(r)(rd\theta) = \frac{1}{2}r^2 d\theta$ The total Area is $Area = \frac{1}{2}\int_{\theta_1}^{\theta_2} r^2 d\theta$.

For the arc-length, we start with $ds = \sqrt{(dx)^2 + (dy)^2}$ and $\begin{cases} x = rCos\theta \\ y = rSin\theta \end{cases}$

Differentiate the transformation respect to unit less $\theta$. $\begin{cases} dx = r'Cos\theta - rSin\theta \\ dy = r'Sin\theta + rCos\theta \end{cases}$

Square and add then simplify. $ds = \sqrt{r^2 + (r')^2}$ we have $s = \int_{\theta_1}^{\theta 2} \sqrt{r^2 + (r')^2} d\theta$.

For Surface Area, we start with $SA = \int_a^b 2\pi(Radius\ of\ rotation)(Arclength)d(\text{var}iable)$

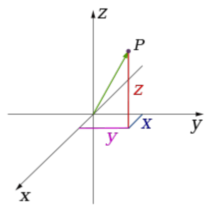
$SA = 2\pi \int_{\theta_1}^{\theta 2} rCos\theta \sqrt{r^2 + (r')^2} d\theta$ Around $\theta = \pi/2$ axes

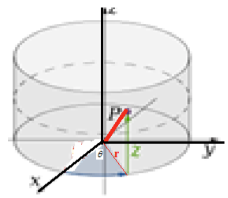
$SA = 2\pi \int_{\theta_1}^{\theta 2} rSin\theta \sqrt{r^2 + (r')^2} d\theta$ Around polar axes

Note that: Slope is the same as rate of change only in Rectangular coordinates.

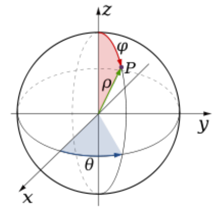
$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$ in parametric coordinates and $\frac{dy}{dx} = \frac{r'Sin\theta + rCos\theta}{r'Cos\theta - rSin\theta}$ in Polar coordinates

Going up one more spatial dimension in polar coordinate, we introduce two other coordinate systems in 3D. (Cylindrical $(r,\theta,z)$, Spherical $(\rho,\theta,\phi)$)



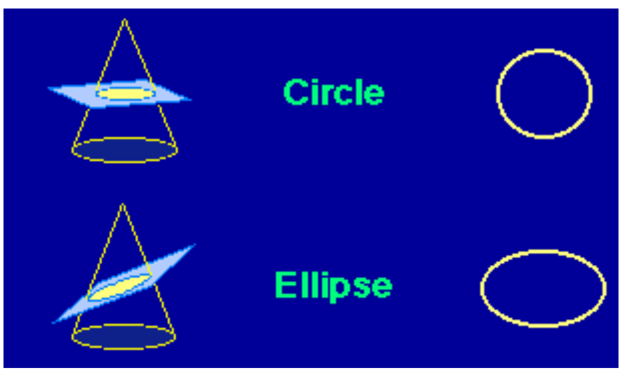
$$\begin{cases} x = rCos\theta \\ y = rSin\theta \\ z = z \end{cases}$$

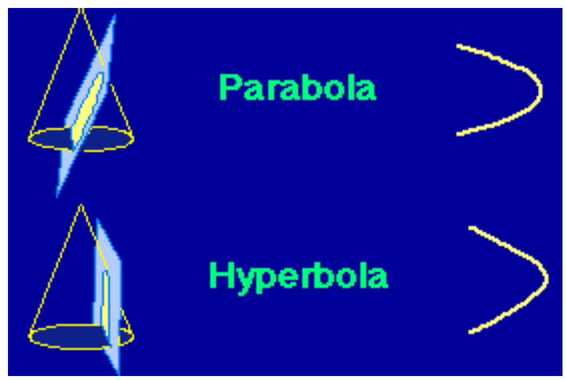



$$\begin{cases} x = \rho Sin\phi Cos\theta \\ y = \rho Sin\phi Sin\theta \\ z = \rho Cos\phi \end{cases}$$

In 3D there are two independent variables $\theta,\phi$ in spherical coordinate. So, $\rho = f(\theta,\phi)$
There are two independent variables $\theta, z$ in cylindrical coordinate. So, $r = f(\theta,z)$
You will learn more about these coordinates in near future.

## Review from Calculus and preCalculus



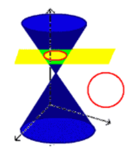

We are living in three-dimensional space and one dimension of time. Unfortunately, it is difficult to imagine the fourth dimension, since it is not visible and most people don't acknowledge it.

In this chapter, I will take you through all these dimensions. We have seen and worked with numbers which are zero dimensions and to us (humans) these are just abstract concepts. Then we learnt about (straight) lines which are one dimension. Two points is needed to define a line since lines have direction (one parameter). Next we go up to 2-D and look at two dimensional shapes. Of course, we know that they don't exist in our (human) three- dimensional space, since they have lack of one dimension but still we can define them mathematically. In this section we will look at two dimensional especial curves which satisfy the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

Imaging two right circular cones are connected with their Vertices. A plane cuts through these combination of cones. Different trace of the plane on the cones is shapes that are caused by the tilt and orientation of the plane. The equation above defines all of these shapes. Let us began with B = 0 to keep our axis horizontal and vertical. (Usual x-y axis)

a) If A = B (the plane is horizontal and the trace of cone on it is a Circle) then we can rewrite the above equation using completing square in to form of

$$\frac{(x-h)^2}{R^2}+\frac{(y-k)^2}{R^2}=1 \text{ where } h=\frac{-D}{2A} \quad k=\frac{-E}{2C} \quad R=\frac{\sqrt{D^2+E^2-4AF}}{2A}$$



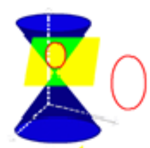
This is equation of a circle with radius R and center at (h , k).

Definition: Location of all points which are equidistance to a fixed point is called **a Circle**.

$R=\sqrt{(x-x_0)^2+(y-y_0)^2}$ For given center $(x_0, y_0)$ and any arbitrary point $(x, y)$

*b)* If A and B are not the same value but they have the same sign (the plane is tilted with the angle between negative 45 to 45 degrees and the trace of the cone on it is an Ellipse) then the equation would be $\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$

$$a=\frac{1}{2A}\sqrt{\frac{D^2C+E^2A-4ACF}{C}} \quad b=\frac{1}{2C}\sqrt{\frac{D^2C+E^2A-4ACF}{A}}$$

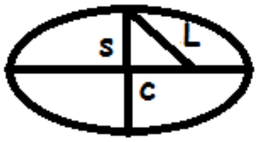


*The tilt of the plane cause the deformation of the shape of the circle to an ellipse, so from the shape of the ellipse one can finds the angle of tilt of the plane.*

Definition: Location of all points which sum of the distances of the point to two fixed points (foci) is equal to 2L, it is called **an Ellipse**.

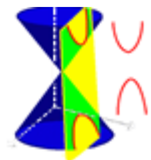
$\sqrt{(x-x_1)^2+(y-y_1)^2}+\sqrt{(x-x_2)^2+(y-y_2)^2}=2L$ where $(x_1, y_1)$ *and* $(x_2, y_2)$ are foci

Eccentricity is the ratio of focal length to length of major axis. It is denoted by $e=\frac{c}{L}$ where c is the distance from center to foci and L is major axis, s is minor axis. The relation among these three can be seen by Pythagorean $L^2=c^2+s^2$




Since c is smaller than L for an ellipse then the eccentricity of an ellipse is less than one $0 \le e < 1$. It is zero in special case for a circle.

c) If A and B have opposite sign (the plane is tilted with angle from 45 to 135 degrees and the trace of the cone on it is a Hyperbola) then we can rewrite the above equation using completing square in to form of $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$



$$a = \frac{1}{2A}\sqrt{\frac{D^2C - E^2A - 4ACF}{C}} \qquad b = \frac{1}{2C}\sqrt{\frac{-D^2C + E^2A + 4ACF}{A}}$$

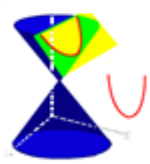
Definition: Location of all points which difference of the distances of the point to two fixed points (foci) is equal to 2L, it is called a **Hyperbola**.

$\sqrt{(x-x_1)^2 + (y-y_1)^2} - \sqrt{(x-x_2)^2 + (y-y_2)^2} = 2L$ where $(x_1, y_1)$ *and* $(x_2, y_2)$ are foci

Eccentricity is the ratio of focal length to length of major axis ($e = \frac{c}{L}$). The relation among L, s, and c is defined by Pythagorean $L^2 + s^2 = c^2$. Since c is larger than L for any hyperbola then the eccentricity of a hyperbola is more than one. $1 < e$

Note: If the Minus sign of the formula is in front of x or y part of the equation then the hyperbola open up in the direction x or y axis.

d) If A or B are not both is zero (the plane is tilted with angle of 45 degrees and the trace of the cone on it is a Pararbola) then we can rewrite the above equation using completing square in to form of $(y-k) = \frac{1}{4p}(x-h)^2$



Where $h = (\frac{-D}{2A})$ , $k = (\frac{D^2 - 4AF}{4AE})$ $p = \frac{-E}{4A}$

Definition: Location of all points which are equi-distance from a fixed line name directrix to a fixed point name focus, it is called a **Pararbola**.

$\sqrt{(x-x_f)^2+(y-y_f)^2}=\sqrt{(x-x)^2+(y-y_d)^2}$ where $(x_f, y_f)$ is focal point and $y=y_d$ is line of directrix. By the above definition c and L are equal.

Eccentricity is the ratio of focal length to length of major axis ($e=\frac{c}{L}$). Since c is equal to L for any Parabola then the eccentricity of a Parabola is equal to one. $e=1$

e) When B is not zero and the cross term (Bxy) present, the axis must be rotated to eliminate the cross term. Following transformation will rotate the axis in two dimensions. $x = X\ Cos\theta - Y\ Sin\theta$ and $y = X\ Sin\theta + Y\ Cos\theta$

The angle of rotation can be found by the following method

1- Find $Tan(2\theta)=\frac{B}{A-C}$

2- Draw a right triangle for angle $2\theta$, and find $Cos(2\theta)$

3- Now $Sin\theta=\sqrt{\frac{1-Cos(2\theta)}{2}}$ and $Cos\theta=\sqrt{\frac{1+Cos(2\theta)}{2}}$

4- Or simply find $Tan\theta=\frac{(C-A)\pm\sqrt{A^2+B^2+C^2-2AC}}{B}$

The x and y coordinate must be replaced by the new coordinate to eliminate B, then follow the methods that we discuss earlier to graph the conic section.

# Parametric equations

Given a horizontal plane parallel to xy- plane respect to x, y, and z coordinate system which can record any trace of an object that falls through it in every one second interval.

*What would be the trace of a Sphere on this plane?* Some concentric circles with radii zero to largest radius of the sphere. The trace starts as a point and grows to a bigger and bigger circle each second.

*What would be the trace of a Cone on this plane?* Some concentric circles with radii zero to the largest radius of the Cone. The trace starts as a point and grows to a bigger and bigger circle each second. You can think of a stationary cone is cut by so many parallel plans which are equally apart (equidistance).

*Is there any difference between these two sets of concentric circles? Is it possible to imagine the shape of the objects that fall through the plane by looking at their trace on the plane only?* The answer is yes, of course. Since the concentric traces made by cylinder, sphere, and cone are different in rate of change their radii. The radius of a cylinder does not change at all, whereas the radius of a cone increases with constant rate if it is inverted and decreases with constant rate if it is right up. In case of sphere, the radius increases with decelerating rate at first until the radius gets as large as possible and then the radius decreases with accelerating rate until it disappears.

Without looking in to the third dimension we can see the third dimension. That is the concept of parametric form of equations. The dimension that can't be seen, parameters are used to define them. You have seen in past that the equation of position and velocity of a particle is in terms of parameter of time since the time dimension can't be seen by human.

Let us introduce a three dimensional space. Including two dependent dimensions of space (x and y axis) and one independent dimension of time which is normal to the plane of x y.

The parametric equations are written in the form of $x = x(t), \quad y = y(t)$. We will look at a few simple equations which described the trace of cone on the plane.

a) Line: The parametric equation of Line is $\begin{cases} x(t) = at + b \\ y(t) = ct + d \end{cases}$ and to convert it in to a rectangular form, one should solve t in terms of on variable and substitute it in to the other variable. $t = \dfrac{x-b}{a}$ and $y = c(\dfrac{x-b}{a}) + d \qquad y = \dfrac{c}{a}x + \dfrac{ad-bc}{a}$

b) Circle: The parametric equation of Circle is $\begin{cases} x(t) = RCos(t) + h \\ y(t) = RSin(t) + k \end{cases}$ and to convert it in to a rectangular form, one should solve for $Cos^2(t) \; and \; Sin^2(t)$ then find the sum

$$Cos^2(t) = \frac{(x-h)^2}{R^2} \qquad Sin^2(t) = \frac{(y-k)^2}{R^2} \quad \text{or} \quad (x-h)^2 + (y-k)^2 = R^2$$

The equation of circle loses some information when it is written in rectangular coordinates. It loses the concept of starting point and direction. In parametric equation of circle at time zero $(x = R + h \;,\; y = k)$ and at $t = \dfrac{\pi}{2}$, $(x = h \;,\; y = R + k)$

These two points gives you the starting point and the direction of rotation C.C.W. One can change the direction by interchanging sine and cosine in the equation.

$$\begin{cases} x(t) = RSin(t) + h \\ y(t) = RCos(t) + k \end{cases}$$

c) Ellipse: The parametric equation of ellipse is $\begin{cases} x(t) = aCos(t) + h \\ y(t) = bSin(t) + k \end{cases}$ and to convert it in to a rectangular form, one should solve for $Cos^2(t)$ *and* $Sin^2(t)$ then find the sum

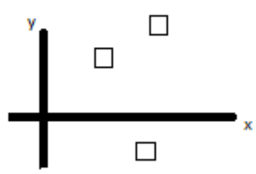
$$Cos^2(t) = \frac{(x-h)^2}{a^2} \quad Sin^2(t) = \frac{(y-k)^2}{b^2} \quad \text{or} \quad \frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

d) Hyperbola: The parametric equation of hyperbola is $\begin{cases} x(t) = aSec(t) + h \\ y(t) = bTan(t) + k \end{cases}$ and to convert it in to a rectangular form, one should solve for $Sec^2(t)$ *and* $Tan^2(t)$ then find the difference $Sec^2(t) = \frac{(x-h)^2}{a^2}, Tan^2(t) = \frac{(y-k)^2}{b^2}$ or $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$

e) Parabola: The parametric equation of parabola is $\begin{cases} x(t) = at + b \\ y(t) = ct^2 + dt + e \end{cases}$

# Polar Coordinates

We have seen and worked with rectangular system where each cell is in a shape of a square (Rectangle). All the rectangles in rectangular coordinates have the same area ( $\Delta A = \Delta x \Delta y$ ). This system is an excellent system for any physical problem that is defined in rectangular (Any geometrical shape with straight edge) box model.

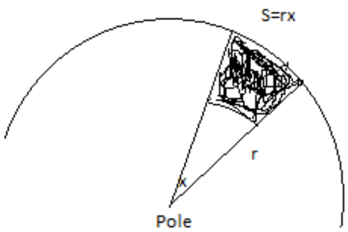



As soon as the nature of the problem changes to the circular shapes, the solution of the problem in rectangular system becomes very un-attractive. A new and better system is needed. There are more than 14 different systems are used in physics which we will look at three of these systems in college level. The two other systems which have the same representation in two dimensions are Cylindrical and Spherical. (The projection of a Sphere and a Cylinder on a flat surface (Plane) is a Circle.)
A Polar coordinate system in 2-dimensions looks like many concentric circles at origin with radii from 0 to N where N is a real number. Any point on coordinate plane will fall on the circumference of one of the circles with a unique radius. So, we know the distance of the point to the origin but there are infinite many points on that circle. To indicate the location of the particular point we introduce the direction or line of sight (line rotated counter clock wise from x-axis with a specific angle).

In this coordinate system, the x-axis is called the polar axis, y-axis is called $\theta = \frac{\pi}{2}$ axis, and the origin is called the pole. The area of each cell in polar coordinates varies according to how far the cell is from the pole with formula $\Delta A = \Delta r.(r\Delta\theta)$ where r is the distance to the pole.

Think of a very small angle $x = \Delta\theta$ and radius $r$ which their product is the arc length of $S = r\Delta\theta$ , if the angle $x = \Delta\theta$ is infinitely small, the shaded region can be seen as a rectangle with length $\Delta r$ and width $r\Delta\theta$ so, the area of the shaded region is $\Delta A = \Delta r(r\Delta\theta)$

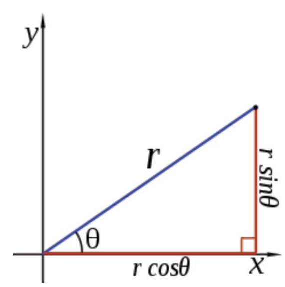
In rectangular coordinates x and y are measured by unit of length therefor the area is in unit of length square. In polar coordinates $r$ is in unit of length but $\Delta\theta$ is in unit of radian and must be multiply by $r$ to make the area in unit of length square. "$r$" in $r\Delta\theta$ is an area correction.




There is a relation from rectangular to polar coordinates since they are isomorphism (every point on one system has an unique image on the other system)

Transformation from Rectangular to Polar

$r = \sqrt{x^2 + y^2} \qquad \theta = Tan^{-1}(\frac{x}{y})$




Transformation from Polar to Rectangular

$x = rCos\theta \qquad y = rSin\theta$

*Plotting points on polar axis:*

1. Draw a line with an angle θ with horizontal (polar) axis.
2. Select *r* on this line
3. Indicate the point with *r* and θ.

*Converting to Polar Equation from Rectangular*

1- Replace x and y in terms of $r \text{ and } \theta$ by the transformation $x = rCos\theta \quad y = rSin\theta$

2- Solve for $r$ in terms of $\theta$

Equation of a Circle: $r^2 = x^2 + y^2$ $r^2 = a^2$ so $r = a$

$x^2 + y^2 = 2Ax$ $r^2 = 2ArCos\theta$ $r = 2ACos\theta$

$x^2 + y^2 = 2Bx$ $r^2 = 2BrSin\theta$ $r = 2BSin\theta$

$x^2 + y^2 = 2Ax + 2By$ $r^2 = 2ArCos\theta + 2BrSin\theta$ $r = 2ACos\theta + 2BSin\theta$

$x^2 - 2Ax + A^2 + y^2 - 2By + B^2 = A^2 + B^2$ $(x-A)^2 + (y-B)^2 = A^2 + B^2$

Center: (A,B) radius: $\sqrt{A^2 + B^2}$

General Equation of Circle: $r^2 = 2ArCos\theta + 2BrSin\theta + C$

Equation of a Line: $y = mx + b$ in polar form $rSin\theta = mrCos\theta + b$ or $r = \dfrac{b}{Sin\theta - mCos\theta}$

$r = bSec\theta$ Vertical Line
$r = bCsc\theta$ Horizontal Line

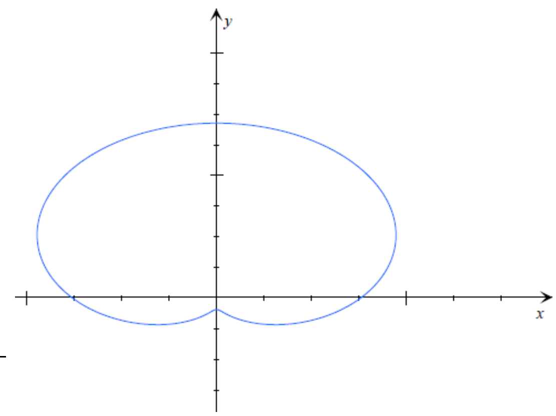
General Equation for a Cardioid

$r = a + bSin\theta$ or $r = a + bCos\theta$

**Case 1** : When a>b

Example $r = 3 + Sin\theta$

| $\theta$ | 0 | $\pi/2$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |
|---|---|---|---|---|---|
| R | 3 | 4 | 3 | 2 | 3 |

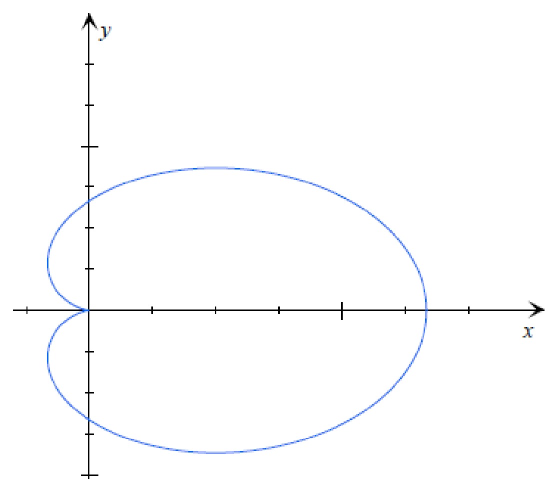



Since r= 3+sin$\theta$ it falls on y-axis(positive)

**Case 2**: when a = b

Example $r = 2 + Cos\theta$

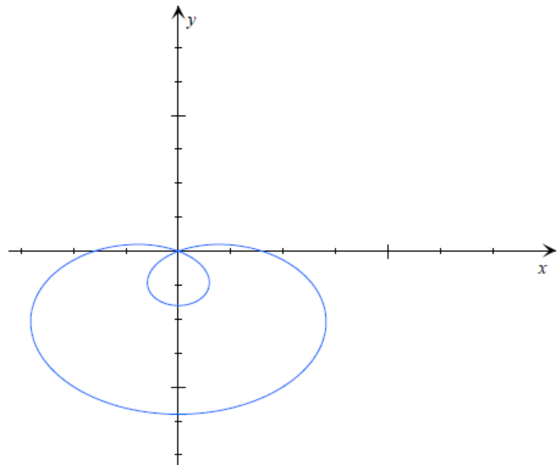
| $\theta$ | 0 | $\pi/2$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |
|---|---|---|---|---|---|
| R | 4 | 2 | 0 | 2 | 4 |



**<u>Case 3</u>:** when a<b

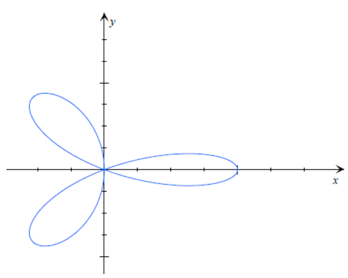
Example $r = 1 - 2Sin\theta$

| $\theta$ | 0 | $\pi/2$ | $\pi$ | $\frac{3\pi}{2}$ | $2\pi$ |
|---|---|---|---|---|---|
| R | 1 | -1 | 1 | 3 | 1 |

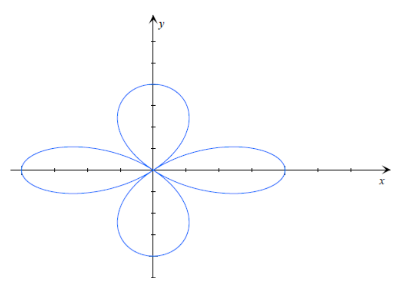



**General equation for <u>rose</u>** $r = 2a\cos(n\theta)$

1. Number of leaves that can be seen $\begin{cases} 2n & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$
2. Angle between leaves $\begin{cases} \frac{360}{2n} \text{ if } n \text{ is even} \\ \frac{360}{n} \text{ if } n \text{ is odd} \end{cases}$
3. Cosine always fall on X axis starting angle 0 or $\pi$.
4. The length of the leaves are 2a




$r = Cos3\theta$

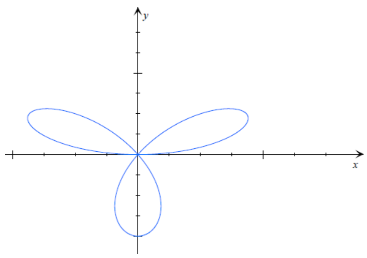



$r = Cos2\theta$

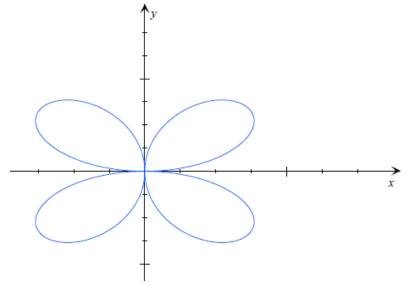
$r = 2a\sin(n\theta)$

- Number of leaves that can be seen $\begin{cases} 2n & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$
- Angle between the leaves $\begin{cases} \frac{360}{2n} & \text{if } n \text{ is even} \\ \frac{360}{n} & \text{if } n \text{ is odd} \end{cases}$
- Starting angle $\frac{90}{n}$

- The length of the leaves are 2a

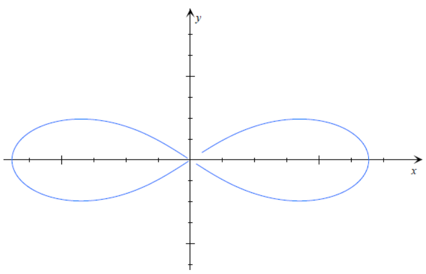


$r = Sin3\theta$

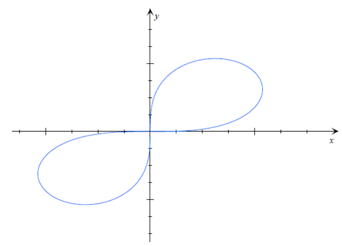


$r = Sin2\theta$

Equation in form of $r^2 = 2\text{acos}\,(2\theta)$ & $r^2 = 2\sin\,(2\theta)$



$r^2 = 2\text{acos}\,(2\theta)$



$r^2 = 2a\sin(2\theta)$

- The dashed curves are imaginary
- Other curves which are not basic curves one must graph them by plotting.

Equation of spirals  r = $\theta$ & r = $\frac{1}{\theta}$

## Polar equations of conics

Ellipse, Parabola, and Hyperbola have the same form of polar equation r = $\frac{eP}{1 \pm ecos\theta}$ it is symmetrical to polar axis and also can be written as r = $\frac{eP}{1 \pm esin\theta}$

e is eccentricity $\begin{cases} e = 0 & circle \\ 0 < e < 1 & ellipse \\ e = 1 & prabola \\ e > 1 & hyperbola \end{cases}$ $|p|$is the distance from pole to the directrix

The steps for graph of $r = \frac{b}{a + cCos\theta}$ or $r = \frac{b}{a + cSin\theta}$ is the same. The difference is in the direction of the graph of conics which opens up in x or y direction. I'll show you the steps for the case that it is in x direction.

Step 1 factor out "a" to find e. $r = \frac{b/a}{1 + (c/a)Cos\theta}$ $e = c/a$

Step 2 Find r when $= 0$ , $\pi/2$ , $\pi$ , $3\pi/2$

Step 3 Plot the points on a polar coordinate axis

Srep4 Decompose $(b/a) = (b/c)(c/a)$ so $b/c = |P|$

Step5 Check eccentricity $e = c/a$ to know the shape of the graph

Step 6 Connect the points

Step 7 Since you know $2a$ , $e = \frac{c}{a}$ , according to $\begin{cases} a^2 - c^2 = b^2 \ Ellipse \\ c^2 - a^2 = b^2 \ Hyperbola \end{cases}$ find b

For ellipse $a^2 = b^2 + (ea)^2$ then $b = a\sqrt{1 - e^2}$ since $e < 1$

For hyperbola $(ea)^2 = a^2 + b^2$ then $b = a\sqrt{e^2 - 1}$ since $e > 1$

What about if the axis was rotated then the polar equation is $r = \frac{ep}{1 \pm eCos(\theta + \alpha)}$ where $\alpha$ the angle of rotation is $\alpha$ . To graph these polar functions, first one ignores the angle $\alpha$ and graph polar function without the rotation angle. Then rotate the axis C.C.W as much as the rotation angle. If the angle is negative the rotation is C.W

The graph of conic sections are either closed $0 \le e < 1$ (Circle, ellipse) which are good models for orbit of planets around a star, or open $1 \le e$ (parabola, Hyperbola)

When eccentricity of path of an object, is a little bit more than one, then the object speeds up as it goes around the foci and the object will throw out from the orbit. (Throw out: Hyper)